# 37.7 Length Contraction

We've seen that relativity requires us to rethink our idea of time. Now let's turn our attention to the concepts of space and distance. Consider the rocket that traveled from the sun to Saturn in Example 37.5. **FIGURE 37.25a** shows the rocket moving with velocity $v$ through the solar system reference frame S. We define $L = \Delta x = x_{\text{Saturn}} - x_{\text{sun}}$ as the distance between the sun and Saturn in frame S or, more generally, the *length* of the spatial interval between two points. The rocket's speed is $v = L/\Delta t$, where $\Delta t$ is the time measured in frame S for the journey from the sun to Saturn.

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**FIGURE 37.25** $L$ and $L'$ are the distances between the sun and Saturn in frames S and S′.

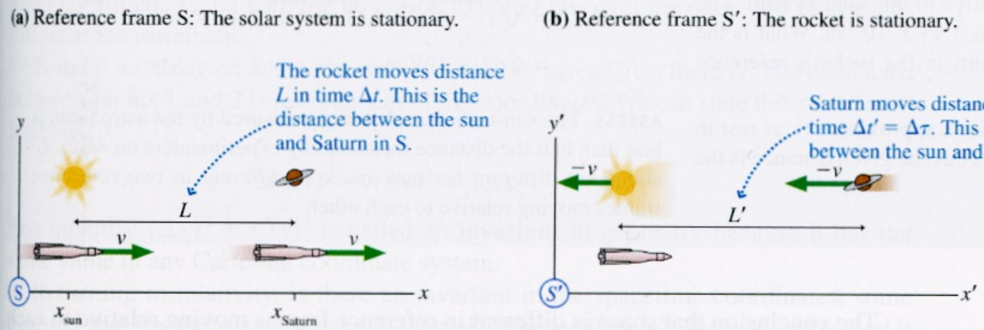



**FIGURE 37.25b** shows the situation in reference frame S′, where the rocket is at rest. The sun and Saturn move to the left at speed $v = L'/\Delta t'$, where $\Delta t'$ is the time measured in frame S′ for Saturn to travel distance $L'$.

Speed $v$ is the relative speed between S and S′ and is the same for experimenters in both reference frames. That is,

$$v = \frac{L}{\Delta t} = \frac{L'}{\Delta t'} \tag{37.12}$$

The time interval $\Delta t'$ measured in frame S′ is the proper time $\Delta\tau$ because both events occur at the same position in frame S′ and can be measured by one clock. We can use the time-dilation result, Equation 37.9, to relate $\Delta\tau$ measured by the astronauts to $\Delta t$ measured by the earthbound scientists. Then Equation 37.12 becomes

$$\frac{L}{\Delta t} = \frac{L'}{\Delta\tau} = \frac{L'}{\sqrt{1-\beta^2}\,\Delta t} \tag{37.13}$$

The $\Delta t$ cancels, and the distance $L'$ in frame S′ is

$$L' = \sqrt{1-\beta^2}\,L \tag{37.14}$$

Surprisingly, we find that **the distance between two objects in reference frame S′ is *not the same* as the distance between the same two objects in reference frame S.**

Frame S, in which the distance is $L$, has one important distinction. It is the *one and only* inertial reference frame in which the objects are at rest. Experimenters in frame S can take all the time they need to measure $L$ because the two objects aren't going anywhere. The distance $L$ between two objects, or two points on one object, measured in the reference frame in which the objects are at rest is called the **proper length** $\ell$. Only one inertial reference frame can measure the proper length.

We can use the proper length $\ell$ to write Equation 37.14 as

$$L' = \sqrt{1-\beta^2}\,\ell \le \ell \tag{37.15}$$

The Stanford Linear Accelerator (SLAC) is a 2-mi-long electron accelerator. The accelerator's length is less than 1 m in the reference frame of the electrons.

This "shrinking" of the distance between two objects, as measured by an experiment moving with respect to the objects, is called **length contraction.** Although we derived length contraction for the distance between two distinct objects, it applies equally well to the length of any physical object that stretches between two points along the $x$- and $x'$-axes. The length of an object is greatest in the reference frame in which the object is at rest. The object's length is less (i.e., the length is contracted) when it is measured in any reference frame in which the object is moving.

**EXAMPLE 37.6 The distance from the sun to Saturn**

In Example 37.5 a rocket traveled along a line from the sun to Saturn at a constant speed of $0.9c$ relative to the solar system. The Saturn-to-sun distance was given as $1.43 \times 10^{12}$ m. What is the distance between the sun and Saturn in the rocket's reference frame?

**MODEL** Saturn and the sun are, at least approximately, at rest in the solar system reference frame S. Thus the given distance is the proper length $\ell$.

**SOLVE** We can use Equation 37.15, to find the distance in the rocket's frame S':

$$L' = \sqrt{1-\beta^2}\,\ell = \sqrt{1-0.9^2}\,(1.43 \times 10^{12}\text{ m})$$
$$= 0.62 \times 10^{12}\text{ m}$$

**ASSESS** The sun-to-Saturn distance measured by the astronauts is less than half the distance measured by experimenters on earth. $L'$ and $\ell$ are different because *space is different* in two reference frames moving relative to each other.

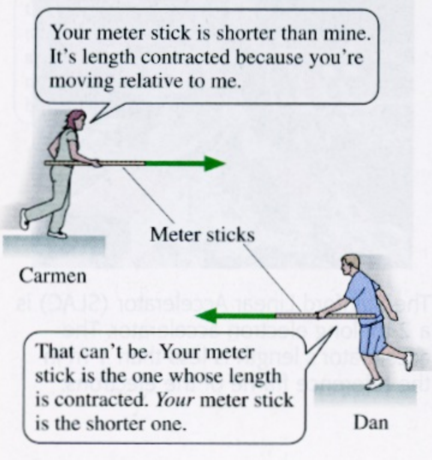
The conclusion that space is different in reference frames moving relative to each other is a direct consequence of the fact that time is different. Experimenters in both reference frames agree on the relative velocity $v$, leading to Equation 37.12: $v = L/\Delta t = L'/\Delta t'$. We had already learned that $\Delta t' < \Delta t$ because of time dilation. Thus $L'$ *has* to be less than $L$. That is the only way experimenters in the two reference frames can reconcile their measurements.

To be specific, the earthly experimenters in Examples 37.5 and 37.6 find that the rocket takes 5300 s to travel the $1.43 \times 10^{12}$ m between the sun and Saturn. The rocket's speed is $v = L/\Delta t = 2.7 \times 10^8$ m/s $= 0.9c$. The astronauts in the rocket find that it takes only 2310 s for Saturn to reach them after the sun has passed by. But there's no conflict, because they also find that the distance is only $0.62 \times 10^{12}$ m. Thus Saturn's speed toward them is $v = L'/\Delta t' = (0.62 \times 10^{12}\text{ m})/(2310\text{ s}) = 2.7 \times 10^8$ m/s $= 0.9c$.

## Another Paradox?

Carmen and Dan are in their physics lab room. They each select a meter stick, lay the two side by side, and agree that the meter sticks are exactly the same length. Then, for an extra-credit project, they go outside and run past each other, in opposite directions, at a relative speed $v = 0.9c$. FIGURE 37.26 shows their experiment and a portion of their conversation.

**FIGURE 37.26** Carmen and Dan each measure the length of the other's meter stick as they move relative to each other.




Now, Dan's meter stick can't be both longer and shorter than Carmen's meter stick. Is this another paradox? No! Relativity allows us to compare the *same* events as they're measured in two different reference frames. This did lead to a real paradox when Peggy rolled past Ryan on the train. There the signal light on the box turns green (a single event) or it doesn't, and Peggy and Ryan have to agree about it. But the events by which Dan measures the length (in Dan's frame) of Carmen's meter stick are *not the same events* as those by which Carmen measures the length (in Carmen's frame) of Dan's meter stick.

There's no conflict between their measurements. In Dan's reference frame, Carmen's meter stick has been length contracted and is less than 1 m in length. In Carmen's reference frame, Dan's meter stick has been length contracted and is less than 1 m in length. If this weren't the case, if both agreed that one of the meter sticks was shorter than the other, then we could tell which reference frame was "really" moving and which was "really" at rest. But the principle of relativity doesn't allow us to make that distinction. Each is moving relative to the other, so each should make the same measurement for the length of the other's meter stick.

## The Spacetime Interval

Forget relativity for a minute and think about ordinary geometry. **FIGURE 37.27** shows two ordinary coordinate systems. They are identical except for the fact that one has been rotated relative to the other. A student using the $xy$-system would measure coordinates $(x_1, y_1)$ for point 1 and $(x_2, y_2)$ for point 2. A second student, using the $x'y'$-system, would measure $(x'_1, y'_1)$ and $(x'_2, y'_2)$.

The students soon find that none of their measurements agree. That is, $x_1 \neq x'_1$ and so on. Even the intervals are different: $\Delta x \neq \Delta x'$ and $\Delta y \neq \Delta y'$. Each is a perfectly valid coordinate system, giving no reason to prefer one over the other, but each yields different measurements.

Is there *anything* on which the two students can agree? Yes, there is. The distance $d$ between points 1 and 2 is independent of the coordinates. We can state this mathematically as

$$d^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2 \qquad (37.16)$$

The quantity $(\Delta x)^2 + (\Delta y)^2$ is called an **invariant** in geometry because it has the same value in any Cartesian coordinate system.

Returning to relativity, is there an invariant in the spacetime coordinates, some quantity that has the *same value* in all inertial reference frames? There is, and to find it let's return to the light clock of Figure 37.21. **FIGURE 37.28** shows the light clock as seen in reference frames S′ and S″. The speed of light is the same in both frames, even though both are moving with respect to each other and with respect to the clock.

Notice that the clock's height $h$ is common to both reference frames. Thus

$$h^2 = \left(\frac{1}{2}c\Delta t'\right)^2 - \left(\frac{1}{2}\Delta x'\right)^2 = \left(\frac{1}{2}c\Delta t''\right)^2 - \left(\frac{1}{2}\Delta x''\right)^2 \qquad (37.17)$$

The factor $\frac{1}{2}$ cancels, allowing us to write

$$c^2(\Delta t')^2 - (\Delta x')^2 = c^2(\Delta t'')^2 - (\Delta x'')^2 \qquad (37.18)$$

Let us define the **spacetime interval** $s$ between two events to be

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2 \qquad (37.19)$$

What we've shown in Equation 37.18 is that **the spacetime interval $s$ has the same value in all inertial reference frames.** That is, the spacetime interval between two events is an invariant. It is a value that all experimenters, in all reference frames, can agree upon.

**FIGURE 37.27** Distance $d$ is the same in both coordinate systems.

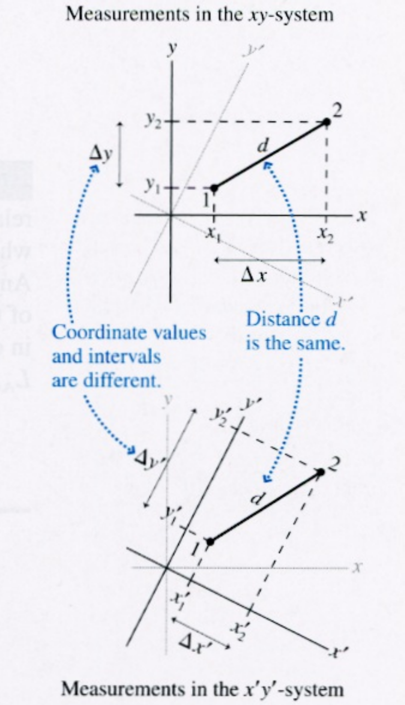
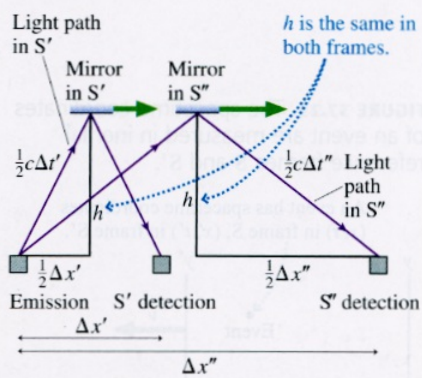



**FIGURE 37.28** The light clock seen by experimenters in reference frames S′ and S″.




**EXAMPLE 37.7 Using the spacetime interval**

A firecracker explodes at the origin of an inertial reference frame. Then, 2.0 μs later, a second firecracker explodes 300 m away. Astronauts in a passing rocket measure the distance between the explosions to be 200 m. According to the astronauts, how much time elapses between the two explosions?

**MODEL** The spacetime coordinates of two events are measured in two different inertial reference frames. Call the reference frame of the ground S and the reference frame of the rocket S′. The spacetime interval between these two events is the same in both reference frames.

**SOLVE** The spacetime interval (or, rather, its square) in frame S is

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2 = (600\text{ m})^2 - (300\text{ m})^2 = 270{,}000\text{ m}^2$$

where we used $c = 300$ m/μs to determine that $c\Delta t = 600$ m. The spacetime interval has the same value in frame S′. Thus

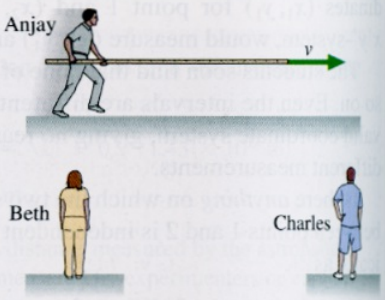
$$s^2 = 270{,}000\text{ m}^2 = c^2(\Delta t')^2 - (\Delta x')^2$$
$$= c^2(\Delta t')^2 - (200\text{ m})^2$$

This is easily solved to give $\Delta t' = 1.85$ μs.

**ASSESS** The two events are closer together in both space and time in the rocket's reference frame than in the reference frame of the ground.

Einstein's legacy, according to popular culture, was the discovery that "everything is relative." But it's not so. Time intervals and space intervals may be relative, as were the intervals $\Delta x$ and $\Delta y$ in the purely geometric analogy with which we opened this section, but some things are *not* relative. In particular, the spacetime interval $s$ between two events is not relative. It is a well-defined number, agreed on by experimenters in each and every inertial reference frame.

**STOP TO THINK 37.7** Beth and Charles are at rest relative to each other. Anjay runs past at velocity $v$ while holding a long pole parallel to his motion. Anjay, Beth, and Charles each measure the length of the pole at the instant Anjay passes Beth. Rank in order, from largest to smallest, the three lengths $L_A$, $L_B$, and $L_C$.




## 37.8 The Lorentz Transformations

The Galilean transformation $x' = x - vt$ of classical relativity lets us calculate the position $x'$ of an event in frame S′ if we know its position $x$ in frame S. Classical relativity, of course, assumes that $t' = t$. Is there a similar transformation in relativity that would allow us to calculate an event's spacetime coordinates $(x', t')$ in frame S′ if we know their values $(x, t)$ in frame S? Such a transformation would need to satisfy three conditions:

1. Agree with the Galilean transformations in the low-speed limit $v \ll c$.
2. Transform not only spatial coordinates but also time coordinates.
3. Ensure that the speed of light is the same in all reference frames.

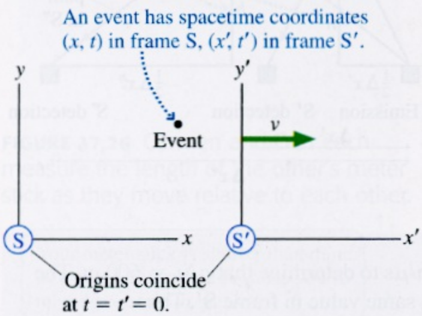



**FIGURE 37.29** The spacetime coordinates of an event are measured in inertial reference frames S and S′.

We'll continue to use reference frames in the standard orientation of **FIGURE 37.29**. The motion is parallel to the $x$- and $x'$-axes, and we *define* $t = 0$ and $t' = 0$ as the instant when the origins of S and S′ coincide.

The requirement that a new transformation agree with the Galilean transformation when $v \ll c$ suggests that we look for a transformation of the form

$$x' = \gamma(x - vt) \quad \text{and} \quad x = \gamma(x' + vt') \tag{37.20}$$

where $\gamma$ is a dimensionless function of velocity that satisfies $\gamma \to 1$ as $v \to 0$.

To determine $\gamma$, we consider the following two events:

Event 1: A flash of light is emitted from the origin of both reference frames $(x = x' = 0)$ at the instant they coincide $(t = t' = 0)$.

Event 2: The light strikes a light detector. The spacetime coordinates of this event are $(x, t)$ in frame S and $(x', t')$ in frame S′.

Light travels at speed $c$ in both reference frames, so the positions of event 2 are $x = ct$ in S and $x' = ct'$ in S′. Substituting these expressions for $x$ and $x'$ into Equation 37.20 gives

$$ct' = \gamma(ct - vt) = \gamma(c - v)t$$
$$ct = \gamma(ct' + vt') = \gamma(c + v)t' \tag{37.21}$$

We solve the first equation for $t'$, by dividing by $c$, then substitute this result for $t'$ into the second:

$$ct = \gamma(c + v)\frac{\gamma(c - v)t}{c} = \gamma^2(c^2 - v^2)\frac{t}{c}$$