Alternatively, suppose we can calculate the probabilities from a theory. In that case, the *expected value* for the number of photons landing in the narrow strip when it is at position x is

$$N(\text{in } \delta x \text{ at } x) = N \times \text{Prob}(\text{in } \delta x \text{ at } x) \tag{40.7}$$

We cannot predict what any individual photon will do, but we can predict the fraction of the photons that should land in this little region of space. Prob(in δx at x) is the probability that it will happen.

40.2 Connecting the Wave and Photon Views

The wave model of light describes the interference pattern in terms of the wave's intensity $I(x)$, a continuous-valued function. The photon model describes the interference pattern in terms of the probability Prob(in δx at x) of detecting a photon. These two models are very different, yet Figure 40.1 shows a clear correlation between the *intensity of the wave* and the *probability of detecting photons.* That is, photons are more likely to be detected at those points where the wave intensity is high and less likely to be detected at those points where the wave intensity is low.

The intensity of a wave is $I = P/A$, the ratio of light power P (joules per second) to the area A on which the light falls. The narrow strip in Figure 40.3a has area $A = H\delta x$. If the light intensity at position x is $I(x)$, the amount of light energy falling onto this narrow strip during each second is

$$E(\text{in } \delta x \text{ at } x) = I(x)A = I(x)H\delta x \tag{40.8}$$

The notation E(in δx at x) refers to the energy landing on this narrow strip if you place it at position x.

From the photon perspective, energy E is due to the arrival of N photons, each of which has energy hf. The number of photons that arrive in the strip each second is

$$N(\text{in } \delta x \text{ at } x) = \frac{E(\text{in } \delta x \text{ at } x)}{hf} = \frac{H}{hf} I(x)\,\delta x \tag{40.9}$$

We can then use the Equation 40.6 definition of probability to write the *probability* that a photon lands in the narrow strip δx at position x as

$$\text{Prob}(\text{in } \delta x \text{ at } x) = \frac{N(\text{in } \delta x \text{ at } x)}{N_{\text{tot}}} = \frac{H}{hfN_{\text{tot}}} I(x)\,\delta x \tag{40.10}$$

Equation 40.10 is a critical link between the wave model and the photon model.

As a final step, recall that the light intensity $I(x)$ is proportional to $|A(x)|^2$, the square of the amplitude function. Consequently,

$$\text{Prob}(\text{in } \delta x \text{ at } x) \propto |A(x)|^2\,\delta x \tag{40.11}$$

where the various constants in Equation 40.10 have all been incorporated into the unspecified proportionality constant of Equation 40.11.

In other words, **the probability of detecting a photon at a particular point is directly proportional to the square of the light-wave amplitude function at that point.** If the wave amplitude at point A is twice that at point B, then a photon is four times as likely to land in a narrow strip at A as it is to land in an equal-width strip at B.

NOTE ► Equation 40.11 is the connection between the particle perspective and the wave perspective. It relates the probability of observing a particle-like event—the arrival of a photon—to the amplitude of a continuous, classical wave. This connection will become the basis of how we interpret the results of quantum-physics calculations. ◄

Probability Density

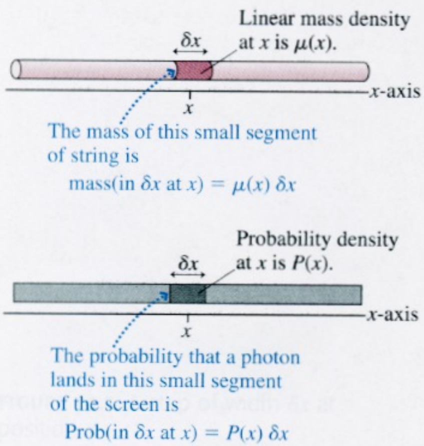
We need one last definition. Recall that the mass of a wire or string of a length L can be expressed in terms of the linear mass density μ as $m = \mu L$. Similarly, the charge along a length L of wire can be expressed in terms of the linear charge density λ as $Q = \lambda L$. If the length had been very short—in which case we might have denoted it as δx,—and if the density varied from point to point, we could have written

$$\text{mass(in length } \delta x \text{ at } x) = \mu(x)\,\delta x$$

$$\text{charge(in length } \delta x \text{ at } x) = \lambda(x)\,\delta x$$

where $\mu(x)$ and $\lambda(x)$ are the linear densities at position x. Writing the mass and charge this way separates the role of the density from the role of the small length δx.

FIGURE 40.4 The probability density is analogous to the linear mass density.



Equation 40.11 looks similar. Using the mass and charge densities as analogies, as shown in FIGURE 40.4, we can define the **probability density** $P(x)$ such that

$$\text{Prob(in } \delta x \text{ at } x) = P(x)\,\delta x \tag{40.12}$$

In one dimension, probability density has SI units of m^{-1}. Thus the probability density multiplied by a length, as in Equation 40.12, yields a dimensionless probability.

NOTE ▶ $P(x)$ itself is *not* a probability, just as the linear mass density λ is not, by itself, a mass. You must multiply the probability density by a length, as shown in Equation 40.12, to find an actual probability. ◀

By comparing Equation 40.12 to Equation 40.11, you can see that the photon probability density is directly proportional to the square of the light-wave amplitude:

$$P(x) \propto |A(x)|^2 \tag{40.13}$$

The probability density, unlike the probability itself, is independent of the width δx and depends on only the position x.

Although we were inspired by the double-slit experiment, nothing in our analysis actually depends on the double-slit geometry. Consequently, Equation 40.13 is quite general. It says that for *any* experiment in which we detect photons, **the probability density for detecting a photon is directly proportional to the square of the amplitude function of the corresponding electromagnetic wave.** We now have an explicit connection between the wave-like and the particle-like properties of the light.

EXAMPLE 40.1 Calculating the probability density

In an experiment, 6000 out of 600,000 photons are detected in a 1.0-mm-wide strip located at position $x = 50$ cm. What is the probability density at $x = 50$ cm?

SOLVE The probability that a photon arrives at this particular strip is

$$\text{Prob(in 1.0 mm at } x = 50\text{ cm}) = \frac{6000}{600{,}000} = 0.010$$

Thus the probability density $P(x) = \text{Prob(in } \delta x \text{ at } x)/\delta x$ at this position is

$$P(50\text{ cm}) = \frac{\text{Prob(in 1.0 mm at } x = 50\text{ cm})}{0.0010\text{ m}} = \frac{0.010}{0.0010\text{ m}}$$
$$= 10\text{ m}^{-1}$$

STOP TO THINK 40.2 The figure shows the detection of photons in an optical experiment. Rank in order, from largest to smallest, the square of the amplitude function of the electromagnetic wave at positions A, B, C, and D.