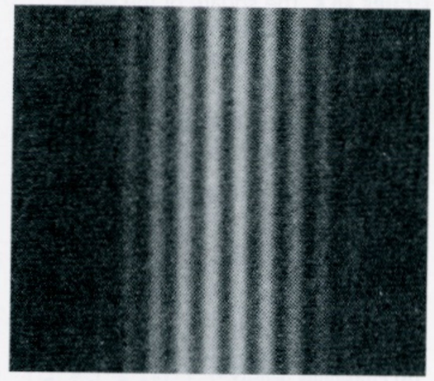
40.3 The Wave Function

Now let's look at the interference of matter. Electrons passing through a double-slit apparatus create the same interference patterns as photons. The pattern is built up electron by electron, but there is no way to predict where any particular electron will be detected. Even so, we can establish the *probability* of an electron landing in a narrow strip of width δx by measuring the positions of many individual electrons.

For light, we were able to relate the photon probability density $P(x)$ to the amplitude of an electromagnetic wave. But there is no wave for electrons like electromagnetic waves for light. So how do we find the probability density for electrons? We have reached the point where we must make an inspired leap beyond classical physics. Let us *assume* that there is some kind of continuous, wave-like function for matter that plays a role analogous to the electromagnetic amplitude function $A(x)$ for light. We will call this function the **wave function** $\psi(x)$, where ψ is a lowercase Greek psi. The wave function is a function of position, which is why we write it as $\psi(x)$.



Electrons create interference fringes after passing through a double slit.

To connect the wave function to the real world of experimental measurements, we will interpret $\psi(x)$ in terms of the *probability* of detecting a particle at position x. If a matter particle, such as an electron, is described by the wave function $\psi(x)$, then the probability Prob(in δx at x) of finding the particle within a narrow region of width δx at position x is

$$\text{Prob(in } \delta x \text{ at } x) = |\psi(x)|^2 \delta x = P(x)\,\delta x \qquad (40.14)$$

That is, the probability density $P(x)$ for finding the particle is

$$P(x) = |\psi(x)|^2 \qquad (40.15)$$

With Equations 40.14 and 40.15, we are *defining* the wave function $\psi(x)$ to play the same role for material particles that the amplitude function $A(x)$ does for photons. The only difference is that $P(x) = |\psi(x)|^2$ is for particles, whereas Equation 40.13 for photons is $P(x) \propto |A(x)|^2$. The difference is because that the electromagnetic field amplitude $A(x)$ had previously been defined through the laws of electricity and magnetism. $|A(x)|^2$ is *proportional* to the probability density for finding a photon, but it is not directly *the* probability density. In contrast, we do not have any preexisting definition for the wave function $\psi(x)$. Thus we are free to define $\psi(x)$ so that $|\psi(x)|^2$ is *exactly* the probability density. That is why we used = rather than $\propto$ in Equation 40.15.

FIGURE 40.5 shows the double-slit experiment with electrons. This time we will work backward. From the observed distribution of electrons, which represents the probabilities of their landing in any particular location, we can deduce that $|\psi(x)|^2$ has alternating maxima and zeros. The oscillatory wave function $\psi(x)$ is the square root *at each point* of $|\psi(x)|^2$. Notice the very close analogy with the amplitude function $A(x)$ in Figure 40.1.

NOTE ► $|\psi(x)|^2$ is uniquely determined by the data, but the wave function $\psi(x)$ is *not* unique. The alternative wave function $\psi'(x) = -\psi(x)$—an upside-down version of the graph in Figure 40.5—would be equally acceptable. ◄

FIGURE 40.6a on the next page is a different example of a wave function. After squaring it *at each point*, as shown in **FIGURE 40.6b**, we see that this wave function represents a particle most likely to be detected very near $x = -b$ or $x = +b$. These are the points where $|\psi(x)|^2$ is a maximum. There is zero likelihood of finding the particle right in the center. The particle is more likely to be detected at some positions than at others, but we cannot predict its exact location.

NOTE ► One of the difficulties in learning to use the concept of a wave function is coming to grips with the fact that there is no "thing" that is waving. There is no disturbance associated with a physical medium. The wave function $\psi(x)$ is simply a *wave-like function* (i.e., it oscillates between positive and negative values) that can be used to make probabilistic predictions about atomic particles. ◄

FIGURE 40.5 The double-slit experiment with electrons.

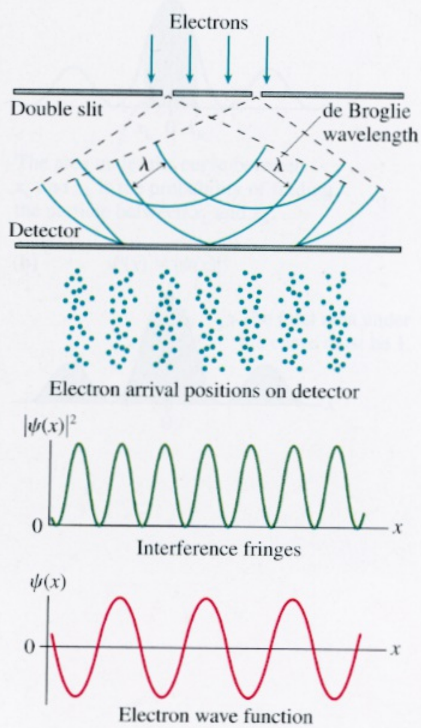
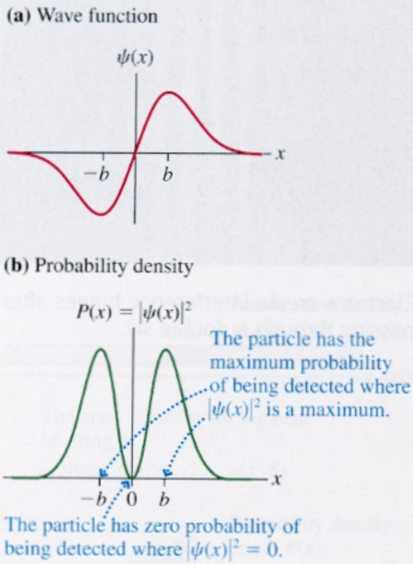


FIGURE 40.6 The square of the wave function is the probability density for detecting the electron at various values of the position x.



A Little Science Methodology

Equation 40.14 defines the wave function $\psi(x)$ for a particle in terms of the probability of finding the particle at different positions x. But our interests go beyond merely characterizing experimental data. We would like to develop a new *theory* of matter. But just what is a theory? Although this is not a book on scientific methodology, we can loosely say that a physical theory needs two basic ingredients:

1. A *descriptor,* a mathematical quantity used to describe our knowledge of a physical object.
2. One or more *laws* that govern the behavior of the descriptor.

For example, Newtonian mechanics is a theory of motion. The primary descriptor in Newtonian mechanics is a particle's *position* $x(t)$ as a function of time. This describes our knowledge of the particle at all times. The position is governed by *Newton's laws.* These laws, especially the second law, are mathematical statements of how the descriptor changes in response to forces. If we predict $x(t)$ for a known set of forces, we feel confident that an experiment carried out at time t will find the particle right where predicted.

Newton's theory of motion *assumes* that a particle's position is well defined at every instant of time. The difficulty facing physicists early in the 20th century was the astounding discovery that **the position of an atomic-size particle is *not* well defined.** An electron in a double-slit experiment must, in some sense, go through *both* slits to produce an electron interference pattern. It simply does not have a well-defined position as it interacts with the slits. But if the position function $x(t)$ is not a valid descriptor for matter at the atomic level, what is?

We will assert that the wave function $\psi(x)$ is the *descriptor* of a particle in quantum mechanics. In other words, the wave function tells us everything we can know about the particle. The wave function $\psi(x)$ plays the same leading role in quantum mechanics that the position function $x(t)$ plays in classical mechanics.

Whether this hypothesis has any merit will not be known until we see if it leads to predictions that can be verified. And before we can do that, we need to learn the "law of psi." What new law of physics determines the wave function $\psi(x)$ in a given situation? We will answer this question in the next chapter.

It may seem to you, as we go along, that we are simply "making up" ideas. Indeed, that is at least partially true. The inventors of entirely new theories use their existing knowledge as a guide, but ultimately they have to make an inspired guess as to what a new theory should look like. Newton and Einstein both made such leaps, and the inventors of quantum mechanics had to make such a leap. We can attempt to make the new ideas *plausible,* but ultimately a new theory is simply a bold new assertion that must be tested against experimental reality. The wave-function theory of quantum mechanics passed the only test that really matters in science—it works!

STOP TO THINK 40.3 This is the wave function of a neutron. At what value of x is the neutron most likely to be found?