40.4 Normalization

In our discussion of probability we noted that the dart has to hit the wall *somewhere*. The mathematical statement of this idea is the requirement that $P_A + P_B + P_C = 1$. That is, the probabilities of all the mutually exclusive outcomes *must* add up to 1.

Similarly, a photon or electron has to land *somewhere* on the detector after passing through an experimental apparatus. Consequently, the probability that it will be detected at *some* position is 100%. To make use of this requirement, consider an experiment in which an electron is detected on the x-axis. As **FIGURE 40.7** shows, we can divide the region between positions x_L and x_R into N adjacent narrow strips of width δx.

The probability that any particular electron lands in the narrow strip i at position x_i is

$$\text{Prob}(\text{in } \delta x \text{ at } x_i) = P(x_i)\,\delta x$$

where $P(x_i) = |\psi(x_i)|^2$ is the probability density at x_i. The probability that the electron lands in the strip at x_1 *or* x_2 *or* x_3 *or* . . . is the sum

$$\begin{aligned}\text{Prob}(\text{between } x_L \text{ and } x_R) &= \text{Prob}(\text{in } \delta x \text{ at } x_1) \\ &\quad + \text{Prob}(\text{in } \delta x \text{ at } x_2) + \cdots \\ &= \sum_{i=1}^{N} P(x_i)\,\delta x = \sum_{i=1}^{N} |\psi(x_i)|^2\,\delta x\end{aligned} \tag{40.16}$$

That is, **the probability that the electron lands *somewhere* between x_L and x_R is the sum of the probabilities of landing in each narrow strip.**

If we let the strips become narrower and narrower, then $\delta x \rightarrow dx$ and the sum becomes an integral. Thus the probability of finding the particles in the range $x_L \leq x \leq x_R$ is

$$\text{Prob}(\text{in range } x_L \leq x \leq x_R) = \int_{x_L}^{x_R} P(x)\,dx = \int_{x_L}^{x_R} |\psi(x)|^2\,dx \tag{40.17}$$

As **FIGURE 40.8a** shows, we can interpret Prob(in range $x_L \leq x \leq x_R$) as the area under the probability density curve between x_L and x_R.

NOTE ▶ The integral of Equation 40.17 is needed when the probability density changes over the range x_L to x_R. For sufficiently narrow intervals, over which $P(x)$ remains essentially constant, the expression Prob(in δx at x) = $P(x)\,\delta x$ is still valid and is easier to use. ◀

Now let the detector become infinitely wide, so that the probability that the electron will arrive *somewhere* on the detector becomes 100%. The statement that the electron has to land *somewhere* on the x-axis is expressed mathematically as

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1 \tag{40.18}$$

Equation 40.18 is called the **normalization condition.** Any wave function $\psi(x)$ must satisfy this condition; otherwise we would not be able to interpret $|\psi(x)|^2$ as a probability density. As **FIGURE 40.8b** shows, Equation 40.18 tells us that the total area under the probability density curve must be 1.

NOTE ▶ The normalization condition integrates the *square* of the wave function. We don't have any information about what the integral of $\psi(x)$ might be. ◀

FIGURE 40.7 Dividing the entire detector into many small strips of width δx.

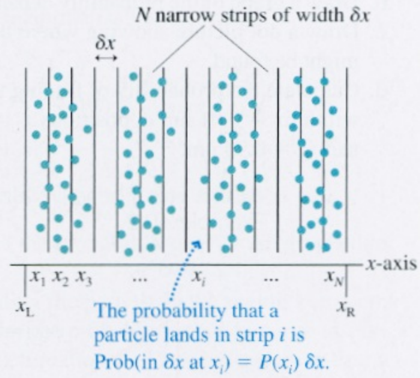
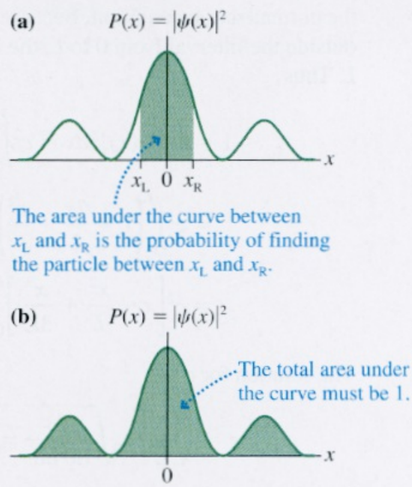


FIGURE 40.8 The area under the probability density curve is a probability.

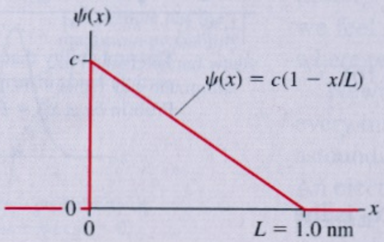


EXAMPLE 40.2 Normalizing and interpreting a wave function

FIGURE 40.9 shows the wave function of a particle confined within the region between $x = 0$ nm and $x = L = 1.0$ nm. The wave function is zero outside this region.

a. Determine the value of the constant c.
b. Draw a graph of the probability density $P(x)$.
c. Draw a dot picture showing where the first 40 or 50 particles might be found.
d. Calculate the probability of finding the particle in a region of width $\delta x = 0.01$ nm at positions $x_1 = 0.05$ nm, $x_2 = 0.50$ nm, and $x_3 = 0.95$ nm.

FIGURE 40.9 The wave function of Example 40.2.



MODEL The probability of finding the particle is determined by the probability density $P(x)$.

VISUALIZE The wave function is shown in Figure 40.9.

SOLVE a. The wave function is $\psi(x) = c(1 - x/L)$ between 0 and L, 0 otherwise. This is a function that decreases linearly from $\psi = c$ at $x = 0$ to $\psi = 0$ at $x = L$. The constant c is the height of this wave function. The particle *has* to be in the region $0 \le x \le L$ with probability 1, and only one value of c will make it so. We can determine c by using Equation 40.18, the normalization condition. Because the wave function is zero outside the interval from 0 to L, the integration limits are 0 to L. Thus

$$1 = \int_0^L |\psi(x)|^2 dx = c^2 \int_0^L \left(1 - \frac{x}{L}\right)^2 dx$$

$$= c^2 \int_0^L \left(1 - \frac{2x}{L} + \frac{x^2}{L^2}\right) dx$$

$$= c^2 \left[x - \frac{x^2}{L} + \frac{x^3}{3L^2}\right]_0^L = \frac{1}{3}c^2 L$$

The solution for c is

$$c = \sqrt{\frac{3}{L}} = \sqrt{\frac{3}{1.0\text{ nm}}} = 1.732\text{ nm}^{-1/2}$$

Note the unusual units for c. Although these are not SI units, we can correctly compute probabilities as long as δx has units of nm. A multiplicative constant such as c is often called a *normalization constant.*

b. The wave function is

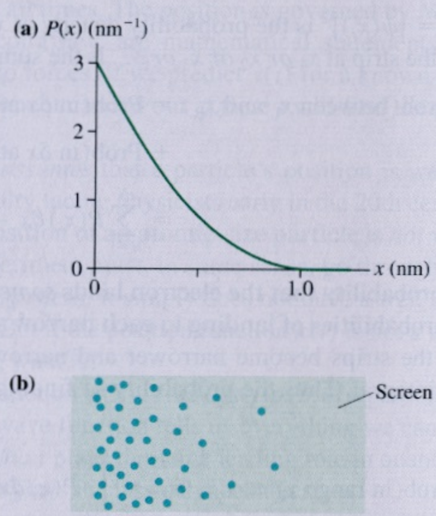
$$\psi(x) = (1.732\text{ nm}^{-1/2})\left(1 - \frac{x}{1.0\text{ nm}}\right)$$

Thus the probability density is

$$P(x) = |\psi(x)|^2 = (3.0\text{ nm}^{-1})\left(1 - \frac{x}{1.0\text{ nm}}\right)^2$$

This probability density is graphed in **FIGURE 40.10a.**

FIGURE 40.10 The probability density $P(x)$ and the detected positions of particles.



c. Particles are most likely to be detected at the left edge of the interval, where the probability density $P(x)$ is maximum. The probability steadily decreases across the interval, becoming zero at $x = 1.0$ nm. **FIGURE 40.10b** shows how a group of particles described by this wave function might appear on a detection screen.

d. $P(x)$ is essentially constant over the small interval $\delta x = 0.01$ nm, so we can use

$$\text{Prob(in } \delta x \text{ at } x) = P(x)\,\delta x = |\psi(x)|^2 \delta x$$

for the probability of finding the particle in a region of width δx at the position x. We need to evaluate $|\psi(x)|^2$ at the three positions $x_1 = 0.05$ nm, $x_2 = 0.50$ nm, and $x_3 = 0.95$ nm. Doing so gives

$$\text{Prob(in 0.01 nm at } x_1 = 0.05\text{ nm}) = c^2(1 - x_1/L)^2 \delta x$$
$$= 0.0270 = 2.70\%$$
$$\text{Prob(in 0.01 nm at } x_2 = 0.50\text{ nm}) = c^2(1 - x_2/L)^2 \delta x$$
$$= 0.0075 = 0.75\%$$
$$\text{Prob(in 0.01 nm at } x_3 = 0.95\text{ nm}) = c^2(1 - x_3/L)^2 \delta x$$
$$= 0.00008 = 0.008\%$$

EXAMPLE 40.3 The probability of finding a particle

A particle is described by the wave function

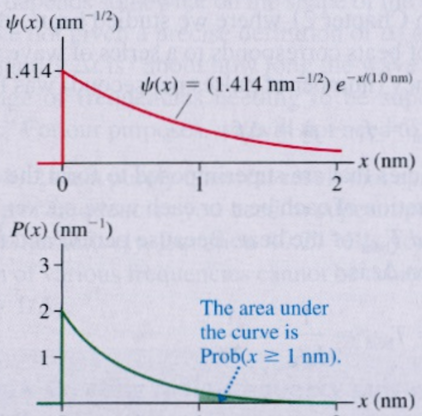
$$\psi(x) = \begin{cases} 0 & x < 0 \\ ce^{-x/L} & x \geq 0 \end{cases}$$

where $L = 1$ nm.

a. Determine the value of the constant c.
b. Draw graphs of $\psi(x)$ and the probability density $P(x)$.
c. Calculate the probability of finding the particle in the region $x \geq 1$ nm.

MODEL The probability of finding the particle is determined by the probability density $P(x)$.

FIGURE 40.11 The wave function and probability density of Example 40.3.



SOLVE a. The wave function is an exponential $\psi(x) = ce^{-x/L}$ that extends from $x = 0$ to $x = +\infty$. Equation 40.18, the normalization condition, is

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = c^2 \int_0^{\infty} e^{-2x/L}\,dx = -\frac{c^2 L}{2} e^{-2x/L}\Big|_0^{\infty} = \frac{c^2}{2L}$$

We can solve this for the normalization constant c:

$$c = \sqrt{\frac{2}{L}} = \sqrt{\frac{2}{1\text{ nm}}} = 1.414\text{ nm}^{-1/2}$$

b. The probability density is

$$P(x) = |\psi(x)|^2 = (2.0\text{ nm}^{-1})e^{-2x/(1.0\text{ nm})}$$

The wave function and the probability density are graphed in **FIGURE 40.11**.

c. The probability of finding the particle in the region $x \geq 1$ nm is the shaded area under the probability density curve in Figure 40.11. We must use Equation 40.17 and integrate to find a numerical value. The probability is

$$\text{Prob}(x \geq 1\text{ nm}) = \int_{1\text{ nm}}^{\infty} |\psi(x)|^2\,dx$$

$$= (2.0\text{ nm}^{-1}) \int_{1\text{ nm}}^{\infty} e^{-2x/(1.0\text{ nm})}\,dx$$

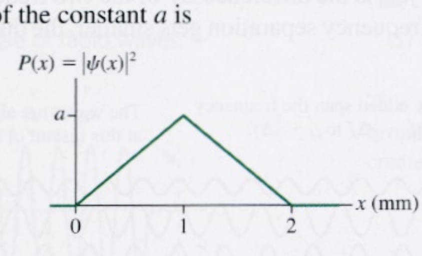
$$= (2.0\text{ nm}^{-1})\left(-\frac{1.0\text{ nm}}{2}\right)e^{-2x/(1.0\text{ nm})}\Big|_{1\text{ nm}}^{\infty}$$

$$= e^{-2} = 0.135 = 13.5\%$$

ASSESS There is a 13.5% chance of finding the particle beyond 1 nm and thus an 86.5% chance of finding it within the interval $0 \leq x \leq 1$ nm. Unlike classical physics, we cannot make an exact prediction of the particle's position.

STOP TO THINK 40.4 The value of the constant a is

a. $a = 2.0\text{ mm}^{-1}$
b. $a = 1.0\text{ mm}^{-1}$
c. $a = 0.5\text{ mm}^{-1}$
d. $a = 2.0\text{ mm}^{-1/2}$
e. $a = 1.0\text{ mm}^{-1/2}$
f. $a = 0.5\text{ mm}^{-1/2}$



40.5 Wave Packets

The classical physics ideas of particles and waves are mutually exclusive. An object can be one or the other, but not both. These classical models fail to describe the wave–particle duality seen at the atomic level. An alternative model with both particle and wave characteristics is a *wave packet*.