EXAMPLE 40.3 The probability of finding a particle

A particle is described by the wave function

$$\psi(x) = \begin{cases} 0 & x < 0 \\ ce^{-x/L} & x \geq 0 \end{cases}$$

where $L = 1$ nm.

a. Determine the value of the constant c.
b. Draw graphs of $\psi(x)$ and the probability density $P(x)$.
c. Calculate the probability of finding the particle in the region $x \geq 1$ nm.

MODEL The probability of finding the particle is determined by the probability density $P(x)$.

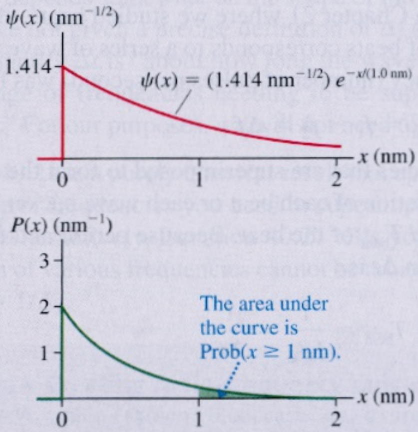


FIGURE 40.11 The wave function and probability density of Example 40.3.

SOLVE a. The wave function is an exponential $\psi(x) = ce^{-x/L}$ that extends from $x = 0$ to $x = +\infty$. Equation 40.18, the normalization condition, is

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = c^2\int_0^{\infty} e^{-2x/L}\,dx = -\frac{c^2L}{2}e^{-2x/L}\Bigg|_0^{\infty} = \frac{c^2}{2L}$$

We can solve this for the normalization constant c:

$$c = \sqrt{\frac{2}{L}} = \sqrt{\frac{2}{1\text{ nm}}} = 1.414\text{ nm}^{-1/2}$$

b. The probability density is

$$P(x) = |\psi(x)|^2 = (2.0\text{ nm}^{-1})e^{-2x/(1.0\text{ nm})}$$

The wave function and the probability density are graphed in **FIGURE 40.11**.

c. The probability of finding the particle in the region $x \geq 1$ nm is the shaded area under the probability density curve in Figure 40.11. We must use Equation 40.17 and integrate to find a numerical value. The probability is

$$\text{Prob}(x \geq 1\text{ nm}) = \int_{1\text{ nm}}^{\infty} |\psi(x)|^2\,dx$$

$$= (2.0\text{ nm}^{-1})\int_{1\text{ nm}}^{\infty} e^{-2x/(1.0\text{ nm})}\,dx$$

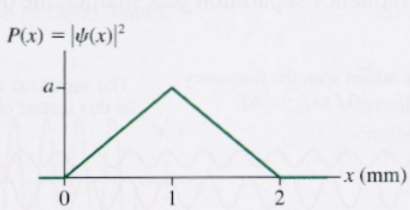
$$= (2.0\text{ nm}^{-1})\left(-\frac{1.0\text{ nm}}{2}\right)e^{-2x/(1.0\text{ nm})}\Bigg|_{1\text{ nm}}^{\infty}$$

$$= e^{-2} = 0.135 = 13.5\%$$

ASSESS There is a 13.5% chance of finding the particle beyond 1 nm and thus an 86.5% chance of finding it within the interval $0 \leq x \leq 1$ nm. Unlike classical physics, we cannot make an exact prediction of the particle's position.

STOP TO THINK 40.4 The value of the constant a is

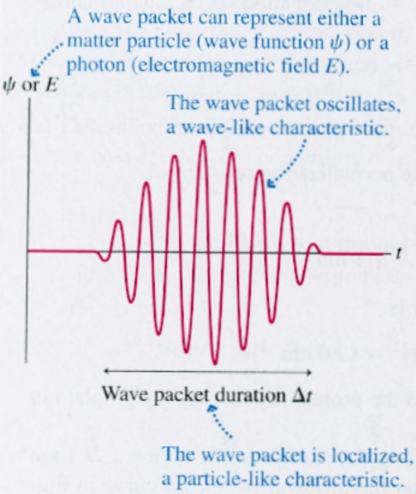
a. $a = 2.0\text{ mm}^{-1}$
b. $a = 1.0\text{ mm}^{-1}$
c. $a = 0.5\text{ mm}^{-1}$
d. $a = 2.0\text{ mm}^{-1/2}$
e. $a = 1.0\text{ mm}^{-1/2}$
f. $a = 0.5\text{ mm}^{-1/2}$



40.5 Wave Packets

The classical physics ideas of particles and waves are mutually exclusive. An object can be one or the other, but not both. These classical models fail to describe the wave–particle duality seen at the atomic level. An alternative model with both particle and wave characteristics is a *wave packet*.

FIGURE 40.12 History graph of a wave packet with duration Δt.



Consider the wave shown in FIGURE 40.12. Unlike the sinusoidal waves we have considered previously, which stretch through time and space, this wave is bunched up, or localized. The localization is a particle-like characteristic. The oscillations are wave-like. Such a localized wave is called a **wave packet.**

A wave packet travels through space with constant speed v, just like a photon in a light wave or an electron in a force-free region. A wave packet has a wavelength, hence it will undergo interference and diffraction. But because it is also localized, a wave packet has the possibility of making a "dot" when it strikes a detector. We can visualize a light wave as consisting of a very large number of these wave packets moving along together. Similarly, we can think of a beam of electrons as a series of wave packets spread out along a line.

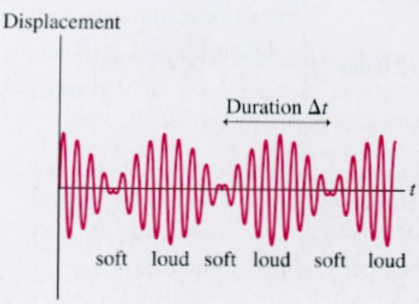
Wave packets are not a perfect model of photons or electrons (we need the full treatment of quantum physics to get a more accurate description), but they do provide a useful way of thinking about photons and electrons in many circumstances.

You might have noticed that the wave packet in Figure 40.12 looks very much like one cycle of a beat pattern. You will recall that beats occur if we superimpose two waves of frequencies f_1 and f_2 where the two frequencies are very similar: $f_1 \approx f_2$. FIGURE 40.13, which is copied from Chapter 21 where we studied beats, shows that the loud, soft, loud, soft, . . . pattern of beats corresponds to a series of wave packets.

In Chapter 21, the beat frequency (number of pulses per second) was found to be

$$f_{\text{beat}} = f_1 - f_2 = \Delta f \tag{40.19}$$

FIGURE 40.13 Beats are a series of wave packets.



where Δf is the *range* of frequencies that are superimposed to form the wave packet. Figure 40.13 defines Δt as the duration of each beat or each wave packet. This interval of time is equivalent to the *period* T_{beat} of the beat. Because period and frequency are inverses of each other, the duration Δt is

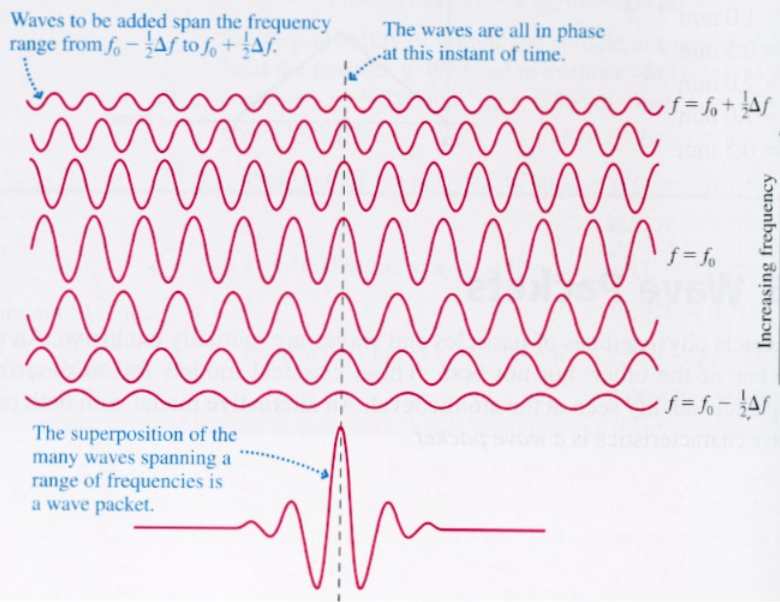
$$\Delta t = T_{\text{beat}} = \frac{1}{f_{\text{beat}}} = \frac{1}{\Delta f}$$

We can rewrite this as

$$\Delta f \Delta t = 1 \tag{40.20}$$

Equation 40.20 is nothing new; we are simply writing what we already knew in a different form. Equation 40.20 is a combination of three things: the relationship $f = 1/T$ between period and frequency, writing T_{beat} as Δt, and the specific knowledge that the beat frequency f_{beat} is the difference Δf of the two frequencies contributing to the wave packet. As the frequency separation gets smaller, the duration of each beat gets longer.

FIGURE 40.14 A single wave packet is the superposition of many component waves of similar wavelength and frequency.



When we superimpose two frequencies to create beats, the wave packet repeats over and over. A more advanced treatment of waves, called Fourier analysis, reveals that a single, *nonrepeating* wave packet can be created through the superposition of *many* waves of very similar frequency. **FIGURE 40.14** illustrates this idea. At one instant of time, all the waves interfere constructively to produce the maximum amplitude of the wave packet. At other times, the individual waves get out of phase and their superposition tends toward zero.

Suppose a single nonrepeating wave packet of duration Δt is created by the superposition of *many* waves that span a range of frequencies Δf. We'll not prove it, but Fourier analysis shows that for *any* wave packet

$$\Delta f \Delta t \approx 1 \qquad (40.21)$$

The relationship between Δf and Δt for a general wave packet is not as precise as Equation 40.20 for beats. There are two reasons for this:

1. Wave packets come in a variety of shapes. The exact relationship between Δf and Δt depends somewhat on the shape of the wave packet.
2. We have not given a precise definition of Δt and Δf for a general wave packet. The quantity Δt is "about how long the wave packet lasts," while Δf is "about the range of frequencies needing to be superimposed to produce this wave packet." For our purposes, we will not need to be any more precise than this.

Equation 40.21 is a purely classical result that applies to waves of any kind. It tells you the range of frequencies you need to superimpose to construct a wave packet of duration Δt. Alternatively, Equation 40.21 tells you that a wave packet created as a superposition of various frequencies cannot be arbitrarily short but *must* last for a time interval $\Delta t \approx 1/\Delta f$.

EXAMPLE 40.4 Creating radio-frequency pulses

A short-wave radio station broadcasts at a frequency of 10.000 MHz. What is the range of frequencies of the waves that must be superimposed to broadcast a radio-wave pulse lasting 0.800 μs?

MODEL A pulse of radio waves is an electromagnetic wave packet, hence it must satisfy the relationship $\Delta f \Delta t \approx 1$.

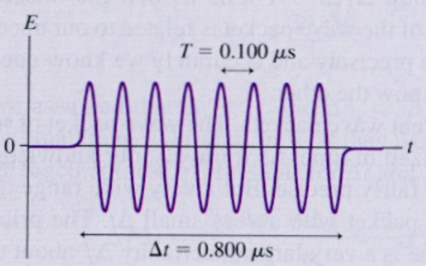


FIGURE 40.15 A pulse of radio waves.

VISUALIZE **FIGURE 40.15** shows the pulse.

SOLVE The period of a 10.000 MHz oscillation is 0.100 μs. A pulse 0.800 μs in duration is 8 oscillations of the wave. Although the station broadcasts at a nominal frequency of 10.000 MHz, this pulse is not a pure 10.000 MHz oscillation. Instead, the pulse has been created by the superposition of many waves whose frequencies span

$$\Delta f \approx \frac{1}{\Delta t} = \frac{1}{0.800 \times 10^{-6}\ \text{s}} = 1.250 \times 10^{6}\ \text{Hz} = 1.250\ \text{MHz}$$

This range of frequencies will be centered at the 10.000 MHz broadcast frequency, so the waves that must be superimposed to create this pulse span the frequency range

$$9.375\ \text{MHz} \le f \le 10.625\ \text{MHz}$$

Bandwidth

Short-duration pulses, like the one in Example 40.4, are used to transmit digital information. Digital signals are sent over a phone line by brief tone pulses, over satellite links by brief radio pulses like the one in the example, and through optical fibers by brief laser-light pulses. Regardless of the type of wave and the medium through which it travels, any wave pulse must obey the fundamental relationship $\Delta f \Delta t \approx 1$.

Sending data at a higher rate (i.e., more pulses per second) requires that the pulse duration Δt be shorter. But a shorter-duration pulse must be created by the superposition of a *larger* range of frequencies. Thus the medium through which a shorter-duration pulse travels must be physically able to transmit the full range of frequencies.

The range of frequencies that can be transmitted through a medium is called the **bandwidth** Δf_B of the medium. The shortest possible pulse that can be transmitted through a medium is

$$\Delta t_{min} \approx \frac{1}{\Delta f_B} \tag{40.22}$$

A pulse shorter than this would require a larger range of frequencies than the medium can support.

The concept of bandwidth is extremely important in digital communications. A higher bandwidth permits the transmission of shorter pulses and allows a higher data rate. A standard telephone line does not have a very high bandwidth, and that is why a modem is limited to sending data at the rate of roughly 50,000 pulses per second. A 0.80 μs pulse can't be sent over a phone line simply because the phone line won't transmit the range of frequencies that would be needed.

An optical fiber is a high-bandwidth medium. A fiber has a bandwidth $\Delta f_B >$ 1 GHz and thus can transmit laser-light pulses with duration $\Delta t <$ 1 ns. As a result, more than 10^9 pulses per second can be sent along an optical fiber, which is why optical-fiber networks form the backbone of the Internet.

Uncertainty

There is another way of thinking about the time-frequency relationship $\Delta f \Delta t \approx 1$. Suppose you want to determine *when* a wave packet arrives at a specific point in space, such as at a detector of some sort. At what instant of time can you say that the wave packet is detected? When the front edge arrives? When the maximum amplitude arrives? When the back edge arrives? Because a wave packet is spread out in time, there is not a unique and well-defined time t at which the packet arrives. All we can say is that it arrives within some interval of time Δt. We are *uncertain* about the exact arrival time.

Similarly, suppose you would like to know the oscillation frequency of a wave packet. There is no precise value for f because the wave packet is constructed from many waves within a range of frequencies Δf. All we can say is that the frequency is within this range. We are *uncertain* about the exact frequency.

The time-frequency relationship $\Delta f \Delta t \approx 1$ tells us that the uncertainty in our knowledge about the arrival time of the wave packet is related to our uncertainty about the packet's frequency. The more precisely and accurately we know one quantity, the less precisely we will be able to know the other.

FIGURE 40.16 Two wave packets with different Δt.

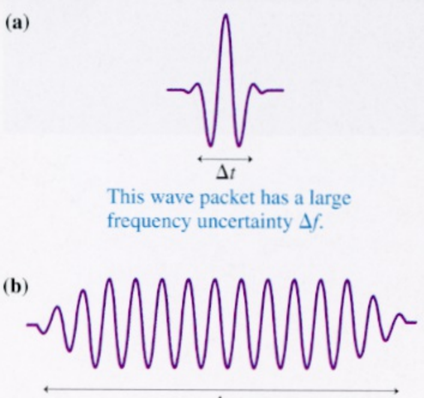


Figure 40.16 shows two different wave packets. The wave packet of **FIGURE 40.16a** is very narrow and thus very localized in time. As it travels, our knowledge of when it will arrive at a specified point is fairly precise. But a very wide range of frequencies Δf is required to create a wave packet with a very small Δt. The price we pay for being fairly certain about the time is a very large uncertainty Δf about the frequency of this wave packet.

FIGURE 40.16b shows the opposite situation: The wave packet oscillates many times and the frequency of these oscillations is pretty clear. Our knowledge of the frequency is good, with minimal uncertainty Δf. But such a wave packet is so spread out that there is a very large uncertainty Δt as to its time of arrival.

In practice, $\Delta f \Delta t \approx 1$ is really a lower limit. Technical limitations may cause the uncertainties in our knowledge of f and t to be even larger than this relationship implies. Consequently, a better statement about our knowledge of a wave packet is

$$\Delta f \Delta t \geq 1 \tag{40.23}$$

The fact that waves are spread out makes it meaningless to specify an exact frequency and an exact arrival time simultaneously. This is an inherent feature of waviness that applies to all waves.

STOP TO THINK 40.5 What minimum bandwidth must a medium have to transmit a 100-ns-long pulse?

a. 1 MHz b. 10 MHz c. 100 MHz d. 1000 MHz

40.6 The Heisenberg Uncertainty Principle

If matter has wave-like aspects and a de Broglie wavelength, then the expression $\Delta f \Delta t \geq 1$ must somehow apply to matter. How? And what are the implications?

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Consider a particle with velocity v_x as it travels along the x-axis with deBroglie wavelength $\lambda = h/p_x$. Figure 40.12 showed a *history graph* (ψ versus t) of a wave packet that might represent the particle as it passes a point on the x-axis. It will be more useful to have a *snapshot graph* (ψ versus x) of the wave packet traveling along the x-axis.

The time interval Δt is the duration of the wave packet as the particle passes a point in space. During this interval, the packet moves forward

$$\Delta x = v_x \Delta t = \frac{p_x}{m}\Delta t \qquad (40.24)$$

where $p_x = mv_x$ is the x-component of the particle's momentum. The quantity Δx, shown in FIGURE 40.17, is the length or spatial extent of the wave packet. Conversely, we can write the wave packet's duration in terms of its length as

$$\Delta t = \frac{m}{p_x}\Delta x \qquad (40.25)$$

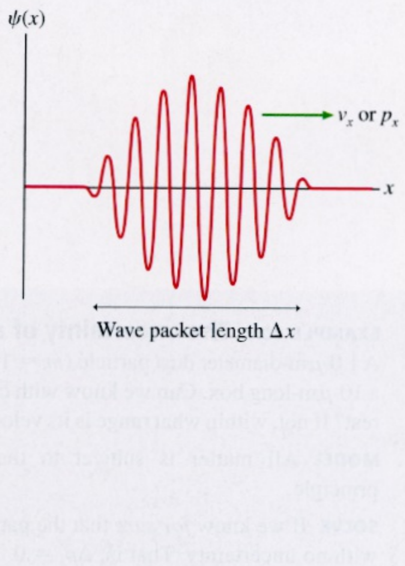

FIGURE 40.17 A snapshot graph of a wave packet.

You will recall that any wave with sinusoidal oscillations must satisfy the wave condition $\lambda f = v$. For a material particle, where λ is the de Broglie wavelength, the frequency f is

$$f = \frac{v}{\lambda} = \frac{p_x/m}{h/p_x} = \frac{p_x^2}{hm}$$

A small range of frequencies Δf is related to a small range of momenta Δp_x by

$$\Delta f = \frac{2p_x \Delta p_x}{hm} \qquad (40.26)$$

where we have assumed that $\Delta f \ll f$ and $\Delta p_x \ll p_x$ (a reasonable assumption) and thus treated the small ranges Δf and Δp_x as if they were differentials df and dp_x.

Multiplying together these expressions for Δt and Δf, we find that

$$\Delta f \Delta t = \frac{2p_x \Delta p_x}{hm}\frac{m\Delta x}{p_x} = \frac{2}{h}\Delta x \Delta p_x \qquad (40.27)$$

Because $\Delta f \Delta t \geq 1$ for any wave, one last rearrangement of Equation 40.27 shows that a matter wave must obey the condition

$$\Delta x \Delta p_x \geq \frac{h}{2} \quad \text{(Heisenberg uncertainty principle)} \qquad (40.28)$$

This statement about the relationship between the position and momentum of a particle was proposed by Werner Heisenberg, creator of one of the first successful quantum theories. Physicists often just call it the **uncertainty principle**.