and an exact arrival time simultaneously. This is an inherent feature of waviness that applies to all waves.

STOP TO THINK 40.5 What minimum bandwidth must a medium have to transmit a 100-ns-long pulse?

a. 1 MHz b. 10 MHz c. 100 MHz d. 1000 MHz

40.6 The Heisenberg Uncertainty Principle

If matter has wave-like aspects and a de Broglie wavelength, then the expression $\Delta f \Delta t \geq 1$ must somehow apply to matter. How? And what are the implications?

ActivPhysics ONLINE 17.6, 17.7

Consider a particle with velocity v_x as it travels along the x-axis with deBroglie wavelength $\lambda = h/p_x$. Figure 40.12 showed a *history graph* (ψ versus t) of a wave packet that might represent the particle as it passes a point on the x-axis. It will be more useful to have a *snapshot graph* (ψ versus x) of the wave packet traveling along the x-axis.

The time interval Δt is the duration of the wave packet as the particle passes a point in space. During this interval, the packet moves forward

$$\Delta x = v_x \Delta t = \frac{p_x}{m}\Delta t \qquad (40.24)$$

where $p_x = mv_x$ is the x-component of the particle's momentum. The quantity Δx, shown in FIGURE 40.17, is the length or spatial extent of the wave packet. Conversely, we can write the wave packet's duration in terms of its length as

$$\Delta t = \frac{m}{p_x}\Delta x \qquad (40.25)$$

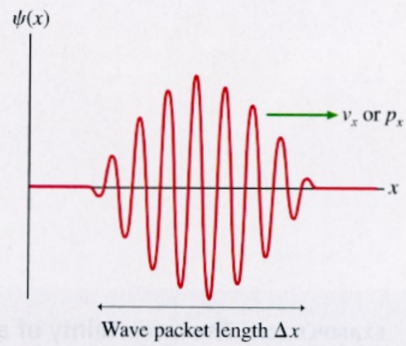

FIGURE 40.17 A snapshot graph of a wave packet.

You will recall that any wave with sinusoidal oscillations must satisfy the wave condition $\lambda f = v$. For a material particle, where λ is the de Broglie wavelength, the frequency f is

$$f = \frac{v}{\lambda} = \frac{p_x/m}{h/p_x} = \frac{p_x^2}{hm}$$

A small range of frequencies Δf is related to a small range of momenta Δp_x by

$$\Delta f = \frac{2p_x \Delta p_x}{hm} \qquad (40.26)$$

where we have assumed that $\Delta f \ll f$ and $\Delta p_x \ll p_x$ (a reasonable assumption) and thus treated the small ranges Δf and Δp_x as if they were differentials df and dp_x.

Multiplying together these expressions for Δt and Δf, we find that

$$\Delta f \Delta t = \frac{2p_x \Delta p_x}{hm}\frac{m\Delta x}{p_x} = \frac{2}{h}\Delta x \Delta p_x \qquad (40.27)$$

Because $\Delta f \Delta t \geq 1$ for any wave, one last rearrangement of Equation 40.27 shows that a matter wave must obey the condition

$$\Delta x \Delta p_x \geq \frac{h}{2} \quad \text{(Heisenberg uncertainty principle)} \qquad (40.28)$$

This statement about the relationship between the position and momentum of a particle was proposed by Werner Heisenberg, creator of one of the first successful quantum theories. Physicists often just call it the **uncertainty principle**.

NOTE ▶ In statements of the uncertainty principle, the right side is sometimes $h/2$, as we have it, but other times it is just h or contains various factors of π. The specific number is not especially important because it depends on exactly how Δx and Δp are defined. The important idea is that the product of Δx and Δp_x for a particle cannot be significantly less than Planck's constant h. A similar relationship for $\Delta y \Delta p_y$ applies along the y-axis. ◀

What Does It Mean?

Heisenberg's uncertainty principle is a statement about our *knowledge* of the properties of a particle. If we want to know *where* a particle is located, we measure its position x. That measurement is not absolutely perfect but has some uncertainty Δx. Likewise, if we want to know *how fast* the particle is going, we need to measure its velocity v_x or, equivalently, its momentum p_x. This measurement also has some uncertainty Δp_x.

Uncertainties are associated with all experimental measurements, but better procedures and techniques can reduce those uncertainties. Newtonian physics places no limits on how small the uncertainties can be. A Newtonian particle at any instant of time has an exact position x and an exact momentum p_x, and with sufficient care we can measure both x and p_x with such precision that the product $\Delta x \Delta p_x \to 0$. There are no inherent limits to our knowledge about a classical, or Newtonian, particle.

Heisenberg, however, made the bold and original statement that our knowledge has real limitations. No matter how clever you are, and no matter how good your experiment, you *cannot* measure both x and p_x simultaneously with arbitrarily good precision. Any measurements you make are limited by the condition that $\Delta x \Delta p_x \geq h/2$. **Our knowledge about a particle is *inherently* uncertain.**

Why? Because of the wave-like nature of matter. The "particle" is spread out in space, so there simply is not a precise value of its position x. Similarly, the de Broglie relationship between momentum and wavelength implies that we cannot know the momentum of a wave packet any more exactly than we can know its wavelength or frequency. Our belief that position and momentum have precise values is tied to our classical concept of a particle. As we revise our ideas of what atomic particles are like, we will also have to revise our old ideas about position and momentum.

EXAMPLE 40.5 The uncertainty of a dust particle

A 1.0-μm-diameter dust particle ($m \approx 10^{-15}$ kg) is confined within a 10-μm-long box. Can we know with certainty if the particle is at rest? If not, within what range is its velocity likely to be found?

MODEL All matter is subject to the Heisenberg uncertainty principle.

SOLVE If we know *for sure* that the particle is at rest, then $p_x = 0$ with no uncertainty. That is, $\Delta p_x = 0$. But then, according to the uncertainty principle, the uncertainty in our knowledge of the particle's position would have to be $\Delta x \to \infty$. In other words, we would have no knowledge at all about the particle's position—it could be anywhere! But that is not the case. We know the particle is *somewhere* in the box, so the uncertainty in our knowledge of its position is at most $\Delta x = L = 10\ \mu$m. With a finite Δx, the uncertainty Δp_x *cannot* be zero. We cannot know with certainty if the particle is at rest inside the box. No matter how hard we try to bring the particle to rest, the uncertainty in our knowledge of the particle's momentum will be $\Delta p_x \approx h/(2\Delta x) = h/2L$. We've assumed the most accurate measurements possible so that the $\geq$ in Heisenberg's uncertainty principle becomes $\approx$. Consequently, the range of possible velocities is

$$\Delta v_x = \frac{\Delta p_x}{m} \approx \frac{h}{2mL} \approx 3.0 \times 10^{-14} \text{ m/s}$$

This range of possible velocities will be centered on $v_x = 0$ m/s if we have done our best to have the particle be at rest. Thus all we can know with certainty is that the particle's velocity is somewhere within the interval -1.5×10^{-14} m/s $\leq v \leq 1.5 \times 10^{-14}$ m/s.

ASSESS For practical purposes you might consider this to be a satisfactory definition of "at rest." After all, a particle moving with a speed of 1.5×10^{-14} m/s would need 6×10^{10} s to move a mere 1 mm. That is about 2000 years! Nonetheless, we can't know if the particle is "really" at rest.

EXAMPLE 40.6 The uncertainty of an electron

What range of velocities might an electron have if confined to a 0.10-nm-wide region, about the size of an atom?

MODEL Electrons are subject to the Heisenberg uncertainty principle.

SOLVE The analysis is the same as in Example 40.5. If we know that the electron's position is located within an interval $\Delta x \approx 0.1$ nm, then the best we can know is that its velocity is within the range

$$\Delta v_x = \frac{\Delta p_x}{m} \approx \frac{h}{2mL} \approx 4 \times 10^6 \text{ m/s}$$

Because the *average* velocity is zero, the best we can say is that the electron's velocity is somewhere in the interval $-2 \times 10^6 \text{ m/s} \le v \le 2 \times 10^6$ m/s. It is simply not possible to know the electron's velocity any more precisely than this.

ASSESS Unlike the situation in Example 40.5, where Δv was so small as to be of no practical consequence, our uncertainty about the electron's velocity is enormous—about 1% of the speed of light!

Once again, we see that even the smallest of macroscopic objects behaves very much like a classical Newtonian particle. Perhaps a 1-μm-diameter particle is slightly fuzzy and has a slightly uncertain velocity, but it is far beyond the measuring capabilities of even the very best instruments to detect this wave-like behavior. In contrast, the effects of the uncertainty principle at the atomic scale are stupendous. We are unable to determine the velocity of an electron in an atom-size container to any better accuracy than about 1% of the speed of light.

STOP TO THINK 40.6 Which of these particles, A or B, can you locate more precisely?