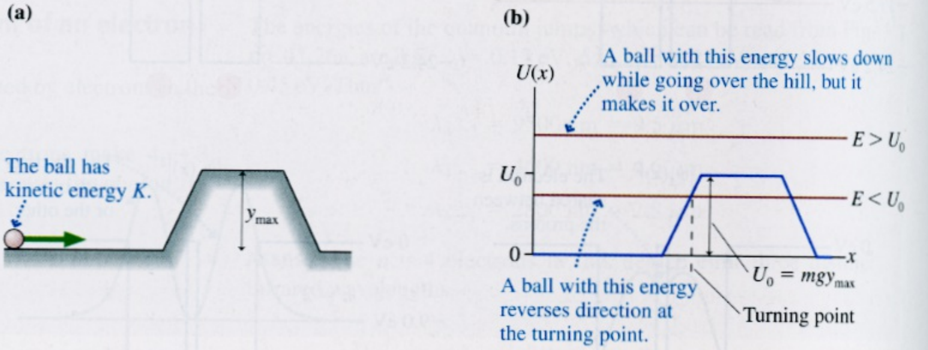
41.10 Quantum-Mechanical Tunneling

20.4 Activ Online Physics

FIGURE 41.29a shows a ball rolling toward a hill. A ball with sufficient kinetic energy can go over the top of the hill, slowing down as it ascends and speeding up as it rolls down the other side. A ball with insufficient energy rolls partway up the hill, then reverses direction and rolls back down.

FIGURE 41.29 A hill is an energy barrier to a rolling ball.



We can think of the hill as an "energy barrier" of height $U_0 = mgy_{max}$. As **FIGURE 41.29b** shows, a ball incident from the left with energy $E > U_0$ can go over the barrier (i.e., roll over the hill), but a ball with $E < U_0$ will reflect from the energy barrier at the turning point. According to the laws of classical physics, a ball that is incident on the energy barrier from the left with $E < U_0$ will never be found on the right side of the barrier.

NOTE ▶ Figure 41.29b is not a "picture" of the energy barrier. And when we say that a ball with energy $E > U_0$ can go "over" the barrier, we don't mean that the ball is thrown from a higher elevation in order to go over the top of the hill. The ball rolls *on the ground* the entire time, as Figure 41.29a shows, and Figure 41.29b describes the kinetic and potential energy of the ball as it rolls. A higher total energy line means a larger initial kinetic energy, not a higher elevation. ◀

FIGURE 41.30 A quantum particle can penetrate through the energy barrier.

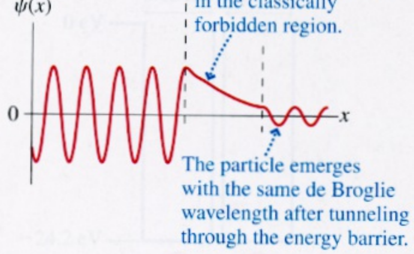


FIGURE 41.30 shows the situation from the perspective of quantum mechanics. As you've learned, quantum particles can penetrate with an exponentially decreasing wave function into the classically forbidden region of an energy barrier. Suppose that the barrier is very narrow. Although the wave function decreases within the barrier, starting at the classical turning point, it hasn't vanished when it reaches the other side. In other words, there is some probability that a quantum particle will pass *through* the barrier and emerge on the other side!

It is very much as if the ball of Figure 41.29a gets to the turning point and then, instead of reversing direction and rolling back down, tunnels its way *through* the hill and emerges on the other side. Although this feat is strictly forbidden in classical mechanics, it is apparently acceptable behavior for quantum particles. The process is called **quantum-mechanical tunneling.**

The process of tunneling through a potential-energy barrier is one of the strangest and most unexpected predictions of quantum mechanics. Yet it does happen, and you will see that it even has many practical applications.

NOTE ▶ The word "tunneling" is used as a metaphor. If a classical particle really did tunnel, it would expend energy doing so and emerge on the other side with less energy. Quantum-mechanical tunneling requires no expenditure of energy. The total energy line is at the same height on both sides of the barrier. A particle that tunnels through a barrier emerges with *no* loss of energy. That is why the de Broglie wavelength is the same on both sides of the potential barrier in Figure 41.30. ◀

To simplify our analysis of tunneling, FIGURE 41.31 shows an idealized energy barrier of height U_0 and width w. We've superimposed the wave function on top of the energy diagram so that you can see how it aligns with the potential energy. The wave function to the left of the barrier is a sinusoidal oscillation with amplitude A_L. The wave function *within* the barrier is the decaying exponential we found in Equation 41.40:

$$\psi_{in}(0 \le x \le w) = \psi_{edge} e^{-x/\eta} = A_L e^{-x/\eta} \qquad (41.51)$$

where we've assumed $\psi_{edge} = A_L$. The penetration distance η was given in Equation 41.41 as

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$

NOTE ▶ You *must* use SI units when calculating values of η. Energies must be in J and $\hbar$ in J s. The penetration distance η has units of meters. ◀

The wave function decreases exponentially within the barrier, but before it can decay to zero, it emerges again on the right side ($x > w$) as an oscillation with amplitude

$$A_R = \psi_{in}(\text{at } x = w) = A_L e^{-w/\eta} \qquad (41.52)$$

The probability that the particle is to the left of the barrier is proportional to $|A_L|^2$, and the probability of finding it to the right of the barrier is proportional to $|A_R|^2$. Thus the probability that a particle striking the barrier from the left will emerge on the right is

$$P_{tunnel} = \frac{|A_R|^2}{|A_L|^2} = (e^{-w/\eta})^2 = e^{-2w/\eta} \qquad (41.53)$$

This is the probability that a particle will tunnel through the energy barrier.

Now, our analysis, we have to say, has not been terribly rigorous. For example, we assumed that the oscillatory wave functions on the left and the right were exactly at a maximum where they reached the barrier at $x = 0$ and $x = w$. There is no reason this has to be the case. We have taken other liberties, which experts will spot, but—fortunately—it really makes no difference. Our result, Equation 41.53, turns out to be perfectly adequate for most applications of tunneling.

Because the tunneling probability is an exponential function, it is *very* sensitive to the values of w and η. The tunneling probability can be substantially reduced by even a small increase in the thickness of the barrier. The parameter η, which measures how far the particle can penetrate into the barrier, depends both on the particle's mass and on $U_0 - E$. A particle with E only slightly less than U_0 will have a larger value of η and thus a larger tunneling probability than will an identical particle with less energy.

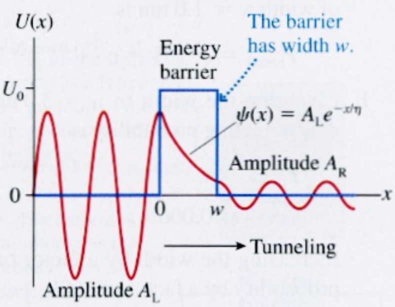


FIGURE 41.31 Tunneling through an idealized energy barrier.

EXAMPLE 41.11 Electron tunneling

a. Find the probability that an electron will tunnel through a 1.0-nm-wide energy barrier if the electron's energy is 0.10 eV less than the height of the barrier.

b. Find the tunneling probability if the barrier in part a is widened to 3.0 nm.

c. Find the tunneling probability if the electron in part a is replaced by a proton with the same energy.

SOLVE a. An electron with energy 0.10 eV less than the height of the barrier has $U_0 - E = 0.10 \text{ eV} = 1.60 \times 10^{-20}$ J. Thus its penetration distance is

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$$
$$= \frac{1.05 \times 10^{-34} \text{ J s}}{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(1.60 \times 10^{-20} \text{ J})}}$$
$$= 6.18 \times 10^{-10} \text{ m} = 0.618 \text{ nm}$$

Continued

The probability that this electron will tunnel through a barrier of width $w = 1.0$ nm is

$$P_{\text{tunnel}} = e^{-2w/\eta} = e^{-2(1.0\text{ nm})/(0.618\text{ nm})} = 0.039 = 3.9\%$$

b. Changing the width to $w = 3.0$ nm has no effect on η. The new tunneling probability is

$$P_{\text{tunnel}} = e^{-2w/\eta} = e^{-2(3.0\text{ nm})/(0.618\text{ nm})} = 6.0 \times 10^{-5}$$
$$= 0.006\%$$

Increasing the width by a factor of 3 decreases the tunneling probability by a factor of 660!

c. A proton is more massive than an electron. Thus a proton with $U_0 - E = 0.10$ eV has $\eta = 0.014$ nm. Its probability of tunneling through a 1.0-nm-wide barrier is

$$P_{\text{tunnel}} = e^{-2w/\eta} = e^{-2(1.0\text{ nm})/(0.014\text{ nm})} \approx 1 \times 10^{-64}$$

For practical purposes, the probability that a proton will tunnel through this barrier is zero.

ASSESS If the probability of a proton tunneling through a mere 1 nm is only 10^{-64}, you can see that a macroscopic object will "never" tunnel through a macroscopic distance!

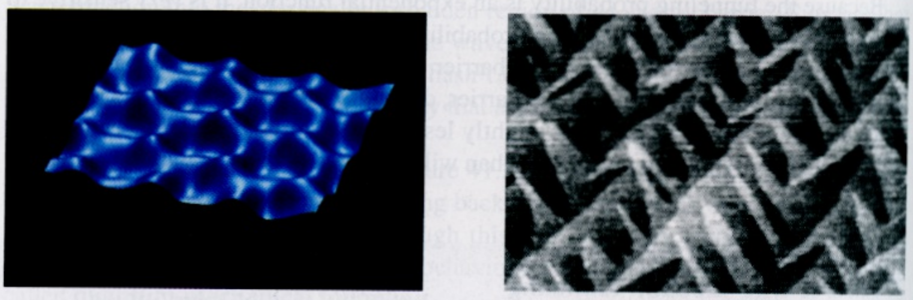
Quantum-mechanical tunneling seems so obscure that it is hard to imagine practical applications. Surprisingly, there are many. We will look at two: the scanning tunneling microscope and the resonant tunneling diode.

The Scanning Tunneling Microscope

Diffraction limits the resolution of an optical microscope to objects no smaller than about a wavelength of light—roughly 500 nm. This is more than 1000 times the size of an atom, so there is no hope of resolving atoms or molecules via optical microscopy. Electron microscopes are similarly limited by the de Broglie wavelength of the electrons. Their resolution is much better than an optical microscope, but still not quite at the level of resolving individual atoms.

This situation changed dramatically in 1981 with the invention of the **scanning tunneling microscope,** or STM as it is usually called. The STM allowed scientists, for the first time, to "see" surfaces literally atom by atom. **FIGURE 41.32** shows two pictures taken with a STM. In one you can see individual atoms of carbon on the surface of graphite. The other shows a somewhat less magnified surface of silicon. These pictures and many others you have likely seen (but may not have known where they came from) are stupendous, but how are they made?

FIGURE 41.32 Two pictures made with a scanning tunneling microscope.



Individual atoms of carbon on the surface of graphite

The surface of silicon

FIGURE 41.33a on the next page shows how the scanning tunneling microscope works. A conducting probe with a *very* sharp tip, just a few atoms wide, is brought to within a few tenths of a nanometer of a surface. Preparing the tips and controlling the spacing are both difficult technical challenges, but scientists have learned how to do both. Once positioned, the probe can mechanically scan back and forth across the surface.

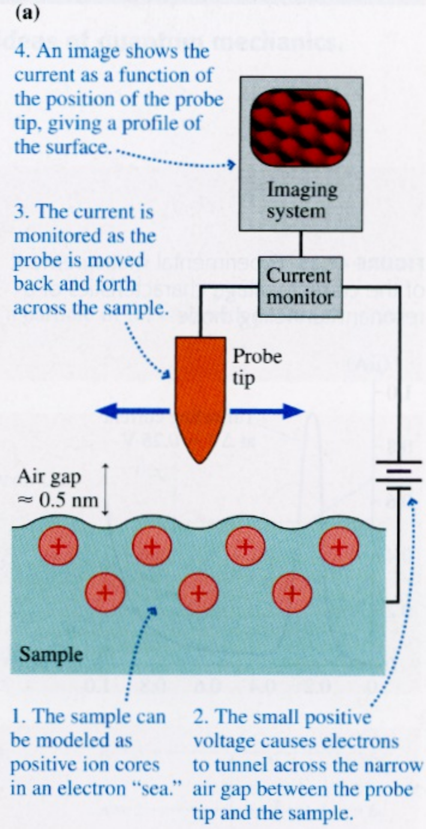
When we analyzed the photoelectric effect, you learned that electrons are bound inside metals by an amount of energy called the *work function* E_0. A typical work function is 4 or 5 eV. This is the energy that must be supplied—by a photon or otherwise—to remove an electron from the metal. In other words, the electron's energy in the metal is E_0 less than its energy outside the metal.

This fact is the basis for the potential-energy diagram of **FIGURE 41.33b**. The small air gap between the sample and the probe tip is a potential-energy barrier. The energy of an electron in the metal of the sample or the probe tip is lower than the energy of an electron in the air by ≈ 4 eV, the work function. The absorption of a photon with $E_{photon} > 4$ eV would lift the electron *over* the barrier, from the sample to the probe. This is just the photoelectric effect. Alternatively, electrons can tunnel *through* the barrier if it is sufficiently narrow. This creates a *tunneling current* from the sample into the probe.

In operation, the tunneling current is recorded as the probe tip scans across the surface. You saw above that the tunneling current is extremely sensitive to the barrier thickness. As the tip scans over the position of an atom, the gap decreases by ≈ 0.1 nm and the current increases. The gap is larger when the tip is between atoms, so the current drops. Today's STMs can sense changes in the gap of as little as 0.001 nm, or about 1% of an atomic diameter! The images you see, such as those in Figure 41.32, are computer-generated from the current measurements at each position.

The STM has revolutionized the science and engineering of microscopic objects. STMs are now used to study everything from how surfaces corrode and oxidize, a topic of great practical importance in engineering, to how biological molecules are structured. Another example of quantum mechanics working for you!

FIGURE 41.33 A scanning tunneling microscope.



The Resonant Tunneling Diode

The semiconductor diode laser that we examined in Section 41.6 had a narrow GaAs layer surrounded by wide layers of GaAlAs. Because an electron's potential energy is ≈ 0.3 eV less in GaAs than in GaAlAs, this structure provides a quantum well in which electrons are confined in a single energy level.

Suppose we manufacture a device in which a thin layer of GaAs is surrounded by still thinner layers of GaAlAs, only a few nanometers thick. **FIGURE 41.34a** is the potential-energy diagram of an electron in such a device. Because the GaAlAs layers are very thin, an electron inside the quantum well can tunnel through to the outside.

Conversely, an electron coming from the outside and impinging on the GaAlAs barrier might tunnel *into* the quantum well. However, tunneling into the well from the outside is hindered by a serious energy mismatch. An electron inside the quantum well *must* have one of the allowed energies. Typically there is a single allowed quantum state with $E_1 \approx 0.15$ eV. Electrons on the outside have thermal energy

$$E_{th} \approx \frac{3}{2}k_B T = 6.0 \times 10^{-21}\ \text{J} = 0.040\ \text{eV}$$

at room temperature. Tunneling may be a strange phenomenon, but energy does still have to be conserved. An electron approaching the barrier with $E \approx 0.04$ eV cannot tunnel inside unless there is a quantum state with this allowed energy.

FIGURE 41.34 Electron potential energy in a resonant tunneling diode.

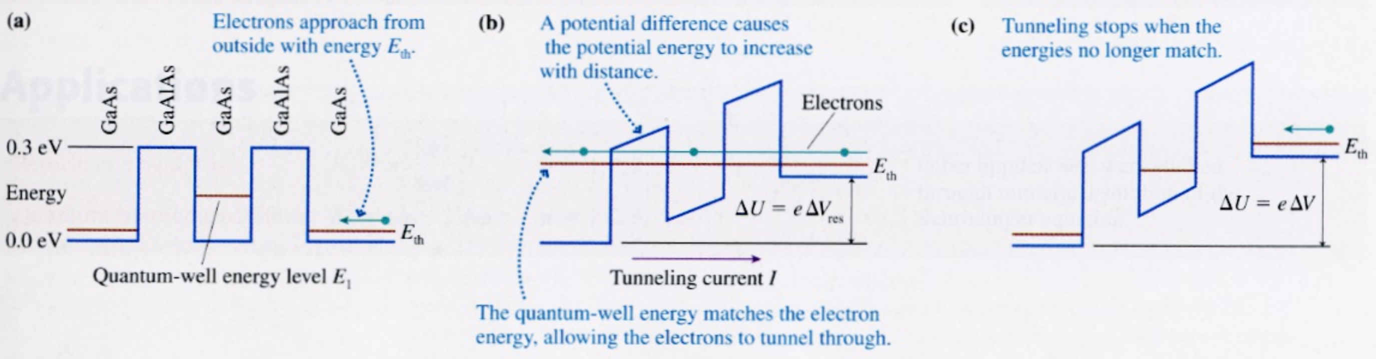


FIGURE 41.34b shows the effect of placing a potential difference ΔV across the three layers of the device. As the potential difference is increased, it will reach a value ΔV_{res} at which the energy level inside the quantum well matches the energy of an electron approaching from the right. We then have a *resonance*, much as when an external driving frequency matches the natural frequency of an oscillator.

Once the energies match, electrons approaching from the right can easily tunnel into the quantum well. They then tunnel through the opposite barrier and emerge on the left with kinetic energy $K \approx e\Delta V$. In other words, there is a current through the device when the potential difference is ΔV_{res}. This device is called a **resonant tunneling diode.**

Too high a voltage destroys the resonance. As **FIGURE 41.34c** shows, a large ΔV drops the energy level in the quantum well too low, so again electrons from the right side have no matching energy level into which they can tunnel. Charge flows through a resonant tunneling diode for only a small range of voltages near ΔV_{res}.

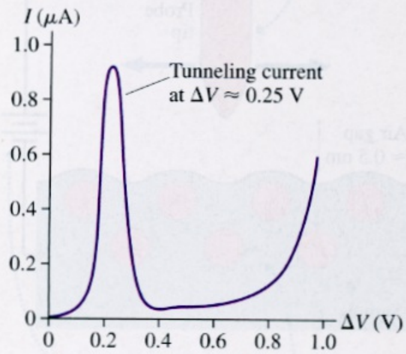


FIGURE 41.35 Experimental measurement of the current-voltage characteristics of a resonant tunneling diode.

FIGURE 41.35 is an experimental current-voltage graph for a device having a 4 nm GaAs quantum well surrounded by 10-nm-wide GaAlAs barriers. There is a small range of voltages around 0.25 volts for which the current shoots up by a factor of 10. This is ΔV_{res}, and the current is due to electrons tunneling through the diode. The current then drops back to near zero by the time $\Delta V = 0.40$ V. (The current increase for $\Delta V > 0.7$ V is "normal" diode behavior. A resonant tunneling diode would not be operated with voltages that large.)

The ability to drastically change current with just a small change in voltage makes tunneling diodes very useful in the digital circuits of high-speed computers. These diodes can also be used as very-high-speed oscillators, creating oscillating voltages with frequencies as high as 500 GHz.

STOP TO THINK 41.6 A particle with energy E approaches an energy barrier with height $U_0 > E$. If U_0 is slowly decreased, the probability that the particle reflects from the barrier

a. Increases.
b. Decreases.
c. Does not change.