EXAMPLE 41.4 The probabilities of locating the particle

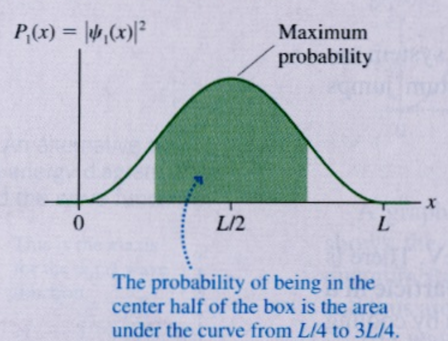
A particle in a rigid box of length L is in its ground state.

a. Where is the particle most likely to be found?
b. What are the probabilities of finding the particle in an interval of width $0.01L$ at $x = 0.00L$, $0.25L$, and $0.50L$?
c. What is the probability of finding the particle in the center half of the box?

MODEL The wave functions for a particle in a rigid box have been determined.

VISUALIZE **FIGURE 41.10** shows the probability density $P_1(x) = |\psi_1(x)|^2$ in the ground state.

FIGURE 41.10 Probability density for a particle in the ground state.



SOLVE a. The particle is most likely to be found at the point where the probability density $P(x)$ is a maximum. You can see from Figure 41.10 that the point of maximum probability for $n = 1$ is $x = L/2$.

b. For a *small* width δx, the probability of finding the particle in δx at position x is

$$\text{Prob(in } \delta x \text{ at } x) = P_1(x)\delta x = |\psi_1(x)|^2 \delta x = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\delta x$$

The interval $\delta x = 0.01L$ is sufficiently small for this to be valid. The probabilities of finding the particle are

$$\text{Prob(in } 0.01L \text{ at } x = 0.00L) = 0.000 = 0.0\%$$

$$\text{Prob(in } 0.01L \text{ at } x = 0.25L) = 0.010 = 1.0\%$$

$$\text{Prob(in } 0.01L \text{ at } x = 0.50L) = 0.020 = 2.0\%$$

c. The center half of the box stretches from $x = L/4$ to $x = 3L/4$. The probability that the particle is in this interval is the area under the probability-density curve:

$$\text{Prob}\left(\text{in interval } \frac{1}{4}L \text{ to } \frac{3}{4}L\right) = \int_{L/4}^{3L/4} P_1(x)\,dx$$

$$= \frac{2}{L}\int_{L/4}^{3L/4} \sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \left[\frac{x}{L} - \frac{1}{\pi}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\right]_{L/4}^{3L/4}$$

$$= \frac{1}{2} + \frac{1}{\pi} = 0.818$$

ASSESS If a particle in a box is in the $n = 1$ ground state, there is an 81.8% chance of finding it in the center half of the box. The probability is greater than 50% because, as you can see in Figure 41.10, the probability density $P_1(x)$ is larger near the center of the box than near the boundaries.

This has been a lengthy presentation of the particle-in-a-box problem. However, it was important that we explore the method of solution completely. Future examples will now go more quickly because many of the issues discussed here will not need to be repeated.

STOP TO THINK 41.2 A particle in a rigid box in the $n = 2$ stationary state is most likely to be found

a. In the center of the box.
b. One-third of the way from either end.
c. One-quarter of the way from either end.
d. It is equally likely to be found at any point in the box.

41.5 The Correspondence Principle

Suppose we confine an electron in a microscopic box, then allow the box to get bigger and bigger. What started out as a quantum-mechanical situation should, when the box becomes macroscopic, eventually look like a classical-physics situation. Similarly, a classical situation such as two charged particles revolving about each other should begin to exhibit quantum behavior as the orbit size becomes smaller and smaller.

These examples suggest that there should be some in-between size, or energy, for which the quantum-mechanical solution corresponds in some way to the solution of classical mechanics. Niels Bohr put forward the idea that the *average* behavior of a quantum system should begin to look like the classical solution in the limit that the quantum number becomes very large—that is, as $n \to \infty$. Because the radius of the Bohr hydrogen atom is $r = n^2 a_B$, the atom becomes a macroscopic object as n becomes very large. Bohr's idea, that the quantum world should blend smoothly into the classical world for high quantum numbers, is today known as the **correspondence principle.**

Our quantum knowledge of a particle in a box is given by its probability density

$$P_{\text{quant}}(x) = |\psi_n(x)|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right) \tag{41.29}$$

To what classical quantity can the probability density be compared as $n \to \infty$?

Interestingly, we can also define a classical probability density $P_{\text{class}}(x)$. A classical particle follows a well-defined trajectory, but suppose we observe the particle at random times. For example, suppose the box containing a classical particle has a viewing window. The window is normally closed, but at random times, selected by a random-number generator, the window opens for a brief interval δt and you can measure the particle's position. When the window opens, what is the probability that the particle will be in a narrow interval δx at position x?

The probability of finding a classical particle within a small interval δx is equal to the *fraction of its time* that it spends passing through δx. That is, you're more likely to find the particle in those intervals δx where it spends lots of time, less likely to find it in a δx where it spends very little time.

If the particle oscillates between two turning points with period T, the time it spends moving from one turning point to the other is $\frac{1}{2}T$. As it moves between the turning points, it passes once through the interval δx at position x, taking time δt to do so. Consequently, the probability of finding the particle within this interval is

$$\text{Prob}_{\text{class}}(\text{in } \delta x \text{ at } x) = \text{fraction of time spent in } \delta x = \frac{\delta t}{\frac{1}{2}T} \tag{41.30}$$

The amount of time needed to pass through δx is $\delta t = \delta x / v(x)$, where $v(x)$ is the particle's velocity at position x. Thus the probability of finding the particle in the interval δx at position x is

$$\text{Prob}_{\text{class}}(\text{in } \delta x \text{ at } x) = \frac{\delta x / v(x)}{\frac{1}{2}T} = \frac{2}{Tv(x)}\delta x \tag{41.31}$$

You learned in Chapter 40 that the probability is related to the probability density by

$$\text{Prob}_{\text{class}}(\text{in } \delta x \text{ at } x) = P_{\text{class}}(x)\,\delta x$$

Thus the classical probability density for finding a particle at position x is

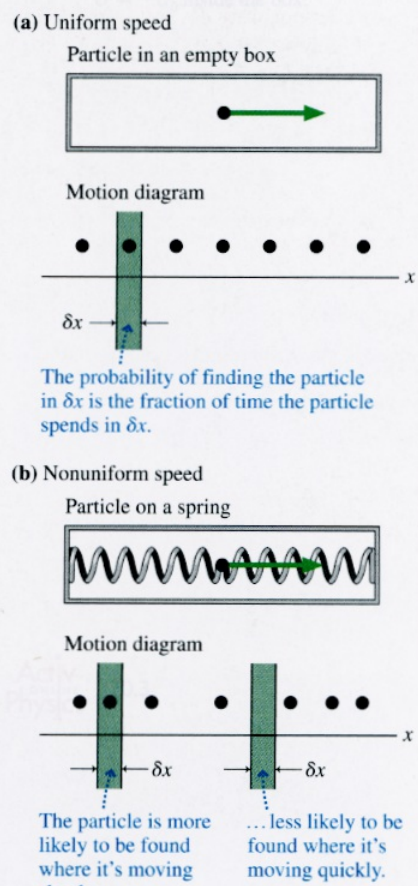
$$P_{\text{class}}(x) = \frac{2}{Tv(x)} \tag{41.32}$$

where the velocity $v(x)$ is expressed as a function of x. **Classically, a particle is more likely to be found where it is moving slowly, less likely to be found where it is moving quickly.**

NOTE ▶ Our derivation of Equation 41.32 made no assumptions about the particle's motion other than the requirement that it be periodic. This is the classical probability density for any oscillatory motion. ◀

FIGURE 41.11a shows a motion diagram of a classical particle in a rigid box of length L. The particle's speed is a *constant* $v(x) = v_0$ as it bounces back and forth between

FIGURE 41.11 The classical probability density is indicated by the density of dots in a motion diagram.



the walls. The particle travels distance $2L$ during one round trip, so the period is $T = 2L/v_0$. Consequently, the classical probability density for a particle in a box is

$$P_{\text{class}}(x) = \frac{2}{(2L/v_0)v_0} = \frac{1}{L} \tag{41.33}$$

$P_{\text{class}}(x)$ is independent of x, telling us that the particle is equally likely to be found *anywhere* in the box.

In contrast, **FIGURE 41.11b** shows a particle with nonuniform speed. A mass on a spring slows down near the turning points, so it spends more time near the ends of the box than in the middle. Consequently the classical probability density for this particle is a maximum at the edges and a minimum at the center. We'll look at this classical probability density again later in the chapter.

EXAMPLE 41.5 The classical probability of locating the particle

A classical particle is in a rigid 10-cm-long box. What is the probability that, at a random instant of time, the particle is in a 1.0-mm-wide interval at the center of the box?

SOLVE The particle's probability density is

$$P_{\text{class}}(x) = \frac{1}{L} = \frac{1}{10\text{ cm}} = 0.10\text{ cm}^{-1}$$

The probability that the particle is in an interval of width $\delta x = 1.0\text{ mm} = 0.10\text{ cm}$ is

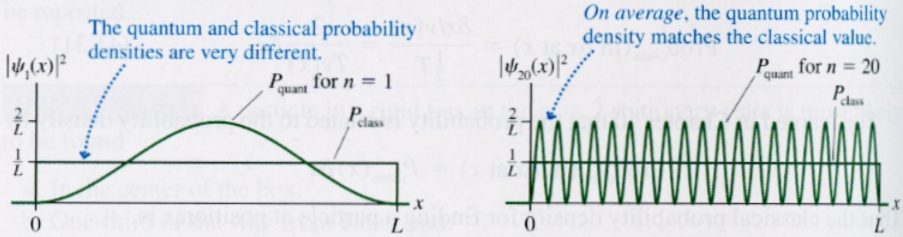
$$\begin{aligned}\text{Prob(in } \delta x \text{ at } x = 5\text{ cm)} &= P(x)\delta x = (0.10\text{ cm}^{-1})(0.10\text{ cm})\\ &= 0.010 = 1.0\%\end{aligned}$$

ASSESS The classical probability is 1.0% because 1.0 mm is 1% of the 10 cm length.

FIGURE 41.12 shows the quantum and the classical probability densities for the $n = 1$ and $n = 20$ quantum states of a particle in a rigid box. Notice that:

- The quantum probability density oscillates between a minimum of 0 and a maximum of $2/L$, so it oscillates around the classical probability density $1/L$.
- For $n = 1$, the quantum and classical probability densities are quite different. The ground state of the quantum system will be very nonclassical.
- For $n = 20$, *on average* the quantum particle's behavior looks very much like that of the classical particle.

FIGURE 41.12 The quantum and classical probability densities for a particle in a box.



As n gets even bigger and the number of oscillations increases, the probability of finding the particle in an interval δx will be the same for both the quantum and the classical particles as long as δx is large enough to include several oscillations of the wave function. As Bohr predicted, the quantum-mechanical solution "corresponds" to the classical solution in the limit $n \rightarrow \infty$.

41.6 Finite Potential Wells

Figure 41.4, the potential-energy diagram for a particle in a rigid box, is an example of a **potential well,** so named because the graph of the potential-energy "hole" looks like a well from which you might draw water. The rigid box was an *infinite* potential well. There was no chance that a particle inside could escape the infinitely high walls.