the walls. The particle travels distance $2L$ during one round trip, so the period is $T = 2L/v_0$. Consequently, the classical probability density for a particle in a box is

$$P_{\text{class}}(x) = \frac{2}{(2L/v_0)v_0} = \frac{1}{L} \tag{41.33}$$

$P_{\text{class}}(x)$ is independent of x, telling us that the particle is equally likely to be found *anywhere* in the box.

In contrast, **FIGURE 41.11b** shows a particle with nonuniform speed. A mass on a spring slows down near the turning points, so it spends more time near the ends of the box than in the middle. Consequently the classical probability density for this particle is a maximum at the edges and a minimum at the center. We'll look at this classical probability density again later in the chapter.

EXAMPLE 41.5 The classical probability of locating the particle

A classical particle is in a rigid 10-cm-long box. What is the probability that, at a random instant of time, the particle is in a 1.0-mm-wide interval at the center of the box?

SOLVE The particle's probability density is

$$P_{\text{class}}(x) = \frac{1}{L} = \frac{1}{10\text{ cm}} = 0.10\text{ cm}^{-1}$$

The probability that the particle is in an interval of width $\delta x = 1.0\text{ mm} = 0.10\text{ cm}$ is

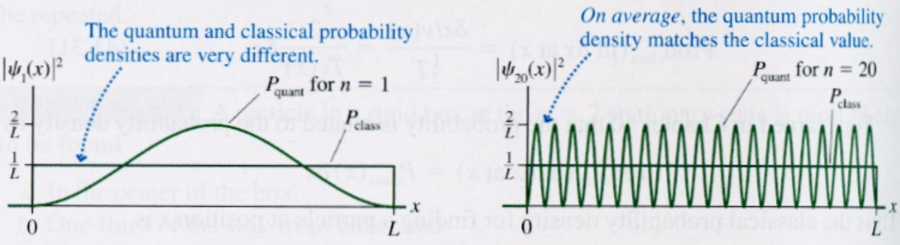
$$\begin{aligned}\text{Prob(in } \delta x \text{ at } x = 5\text{ cm)} &= P(x)\delta x = (0.10\text{ cm}^{-1})(0.10\text{ cm}) \\ &= 0.010 = 1.0\%\end{aligned}$$

ASSESS The classical probability is 1.0% because 1.0 mm is 1% of the 10 cm length.

FIGURE 41.12 shows the quantum and the classical probability densities for the $n = 1$ and $n = 20$ quantum states of a particle in a rigid box. Notice that:

- The quantum probability density oscillates between a minimum of 0 and a maximum of $2/L$, so it oscillates around the classical probability density $1/L$.
- For $n = 1$, the quantum and classical probability densities are quite different. The ground state of the quantum system will be very nonclassical.
- For $n = 20$, *on average* the quantum particle's behavior looks very much like that of the classical particle.

FIGURE 41.12 The quantum and classical probability densities for a particle in a box.



As n gets even bigger and the number of oscillations increases, the probability of finding the particle in an interval δx will be the same for both the quantum and the classical particles as long as δx is large enough to include several oscillations of the wave function. As Bohr predicted, the quantum-mechanical solution "corresponds" to the classical solution in the limit $n \to \infty$.

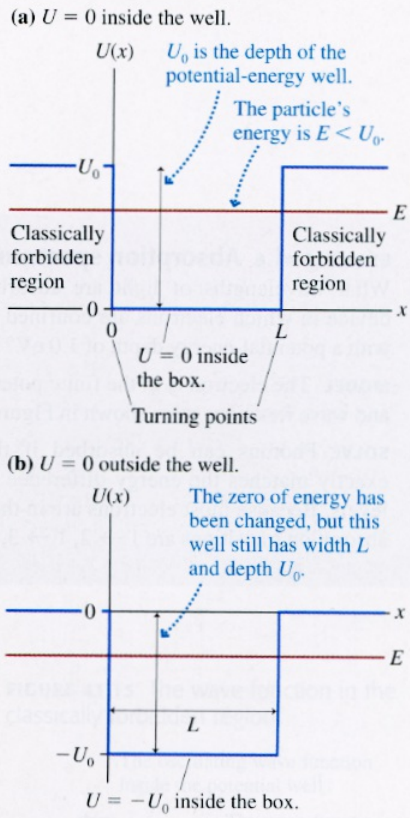
41.6 Finite Potential Wells

Figure 41.4, the potential-energy diagram for a particle in a rigid box, is an example of a **potential well,** so named because the graph of the potential-energy "hole" looks like a well from which you might draw water. The rigid box was an *infinite* potential well. There was no chance that a particle inside could escape the infinitely high walls.

No box is infinitely strong. A more realistic model of a confined particle is the *finite* potential well shown in **FIGURE 41.13a**. A particle with total energy $E < U_0$ is confined within the well, bouncing back and forth between turning points at $x = 0$ and $x = L$. The regions $x < 0$ and $x > L$ are **classically forbidden regions** for a particle with $E < U_0$. However, the particle will escape the well if it somehow manages to acquire energy $E > U_0$.

Recall that the zero of energy is arbitrary. Figure 41.13a defined $U = 0$ as the potential energy inside the well. **FIGURE 41.13b** has repositioned the zero of energy at the level of the "energy plateau" on both sides of the well. **Figures 41.13a and 41.13b are the same potential well.** Both have width L and depth U_0, and both have the same wave functions and the same allowed energies (relative to our choice of $E = 0$). Which we use is a matter of convenience.

FIGURE 41.13 A finite potential well of width L and depth U_0.

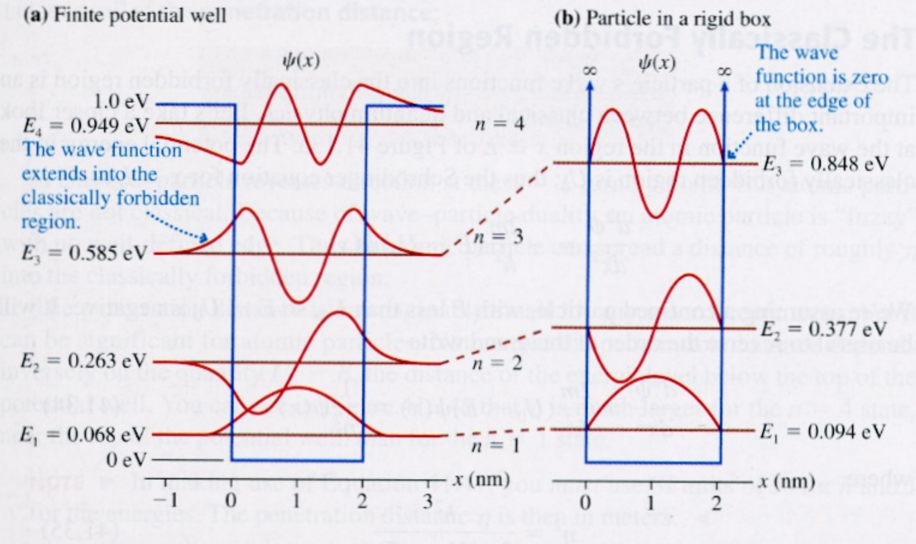


We've made no mention of the *force* that is responsible for this potential well. An electron confined within a semiconductor by an electric force has a potential energy that can be modeled as a finite potential well. So does a proton confined within the nucleus by the nuclear force. The Schrödinger equation depends on the *shape* of the potential-energy function, not the cause. Hence *any* situation in which a particle is confined can be modeled as a finite potential well.

Although it is possible to solve the Schrödinger equation exactly for the finite potential well, the result is cumbersome and not especially illuminating. Instead, we'll present the results of numerical calculations. The derivation of the wave functions and energy levels is not as important as understanding and interpreting the results.

As a first example, consider an electron in a 2.0-nm-wide potential well of depth $U_0 = 1.0$ eV. These are reasonable parameters for an electron in a semiconductor device. **FIGURE 41.14a** is a graphical presentation of the allowed energies and wave functions. For comparison, **FIGURE 41.14b** shows the first three energy levels and wave functions for a rigid box ($U_0 \to \infty$) with the same 2.0 nm width.

FIGURE 41.14 Energy levels and wave functions for a finite potential well. For comparison, the energies and wave functions are shown for a rigid box of equal width.



The quantum-mechanical solution for a particle in a finite potential well has some important properties:

Activ Physics 20.3

- The particle's energy is quantized. A particle in the potential well *must* be in one of the stationary states with quantum numbers $n = 1, 2, 3, \ldots$.
- There are only a finite number of **bound states**—four in this example, although the number will be different in other examples. These wave functions represent electrons confined to, or bound in, the potential well. There are no stationary states with $E > U_0$ because such a particle would not remain in the well.

- The wave functions are qualitatively similar to those of a particle in a rigid box, but the energies are somewhat lower. This is because the wave functions are slightly more spread out. A slightly longer de Broglie wavelength corresponds to a lower velocity and thus a lower energy.
- Most interesting, perhaps, is that the wave functions of Figure 41.14a extend into the classically forbidden regions. It is as though a tennis ball penetrated partly *through* the racket's strings before bouncing back, but without breaking the strings.

EXAMPLE 41.6 Absorption spectrum of an electron

What wavelengths of light are absorbed by a semiconductor device in which electrons are confined in a 2.0-nm-wide region with a potential-energy depth of 1.0 eV?

MODEL The electron is in the finite potential well whose energies and wave functions were shown in Figure 41.14a.

SOLVE Photons can be absorbed if their energy $E_{\text{photon}} = hf$ exactly matches the energy difference ΔE between two energy levels. Because most electrons are in the $n = 1$ ground state, the absorption transitions are $1 \rightarrow 2$, $1 \rightarrow 3$, and $1 \rightarrow 4$.

The absorption wavelengths $\lambda = c/f$ are

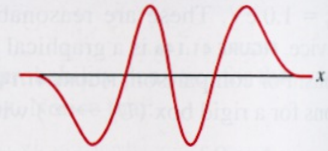
$$\lambda_{n\rightarrow m} = \frac{hc}{\Delta E} = \frac{hc}{|E_n - E_m|}$$

For this example, we find

$$\Delta E_{1\text{-}2} = 0.195\text{ eV} \quad \lambda_{1\rightarrow 2} = 6.37\ \mu\text{m}$$
$$\Delta E_{1\text{-}3} = 0.517\text{ eV} \quad \lambda_{1\rightarrow 3} = 2.40\ \mu\text{m}$$
$$\Delta E_{1\text{-}4} = 0.881\text{ eV} \quad \lambda_{1\rightarrow 4} = 1.41\ \mu\text{m}$$

ASSESS These transitions are all infrared wavelengths.

STOP TO THINK 41.3 This is a wave function for a particle in a finite quantum well. What is the particle's quantum number?



The Classically Forbidden Region

The extension of a particle's wave functions into the classically forbidden region is an important difference between classical and quantum physics. Let's take a closer look at the wave function in the region $x \geq L$ of Figure 41.13a. The potential energy in the classically forbidden region is U_0; thus the Schrödinger equation for $x \geq L$ is

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - U_0)\psi(x)$$

We're assuming a confined particle, with E less than U_0, so $E - U_0$ is negative. It will be useful to reverse the order of these and write

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U_0 - E)\psi(x) = \frac{1}{\eta^2}\psi(x) \tag{41.34}$$

where

$$\eta^2 = \frac{\hbar^2}{2m(U_0 - E)} \tag{41.35}$$

is a *positive* constant. As a homework problem, you can show that the units of η are meters.

The Schrödinger equation of Equation 41.34 is one we can solve by guessing. We simply need to think of two functions whose second derivatives are a positive constant times the functions themselves. Two such functions, as you can quickly confirm, are $e^{x/\eta}$ and $e^{-x/\eta}$. Thus, according to Equation 41.8, the general solution of the Schrödinger equation for $x \geq L$ is

$$\psi(x) = Ae^{x/\eta} + Be^{-x/\eta} \quad \text{for } x \geq L \tag{41.36}$$

One requirement of the wave function is that $\psi \to 0$ as $x \to \infty$. The function $e^{x/\eta}$ diverges as $x \to \infty$, so the only way to satisfy this requirement is to set $A = 0$. Thus

$$\psi(x) = Be^{-x/\eta} \quad \text{for } x \geq L \tag{41.37}$$

This is an exponentially decaying function. Notice that all the wave functions in Figure 41.14a look like exponential decays for $x > L$.

The wave function must also be continuous. Suppose the oscillating wave function within the potential well ($x \leq L$) has the value ψ_{edge} when it reaches the classical boundary at $x = L$. To be continuous, the wave function of Equation 41.37 has to match this value at $x = L$. That is,

$$\psi(\text{at } x = L) = Be^{-L/\eta} = \psi_{\text{edge}} \tag{41.38}$$

This boundary condition at $x = L$ is sufficient to determine that the constant B is

$$B = \psi_{\text{edge}}e^{L/\eta} \tag{41.39}$$

If we use the Equation 41.39 result for B in Equation 41.37, we find that the wave function in the classically forbidden region of a finite potential well is

$$\psi(x) = \psi_{\text{edge}}e^{-(x-L)/\eta} \quad \text{for } x \geq L \tag{41.40}$$

In other words, **the wave function oscillates until it reaches the classical turning point at $x = L$, then it decays exponentially within the classically forbidden region.** A similar analysis could be done for $x \leq 0$.

FIGURE 41.15 shows the wave function in the classically forbidden region. You can see that the wave function at $x = L + \eta$ has decreased to

$$\psi(\text{at } x = L + \eta) = e^{-1}\psi_{\text{edge}} = 0.37\psi_{\text{edge}}$$

Although an exponential decay does not have a sharp ending point, the parameter η measures "about how far" the wave function extends past the classical turning point before the probability of finding the particle has decreased nearly to zero. This distance is called the **penetration distance:**

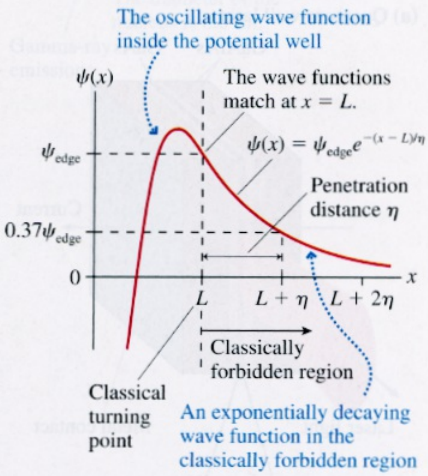
$$\text{penetration distance } \eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} \tag{41.41}$$

A classical particle reverses direction at the $x = L$ turning point. But atomic particles are not classical. Because of wave–particle duality, an atomic particle is "fuzzy" with no well-defined edge. Thus an atomic particle can spread a distance of roughly η into the classically forbidden region.

The penetration distance is unimaginably small for any macroscopic mass, but it can be significant for atomic particles. Notice that the penetration distance depends inversely on the quantity $U_0 - E$, the distance of the energy level below the top of the potential well. You can see in Figure 41.14a that η is much larger for the $n = 4$ state, near the top of the potential well, than for the $n = 1$ state.

NOTE ► In making use of Equation 41.41, you *must* use SI units of J s for $\hbar$ and J for the energies. The penetration distance η is then in meters. ◄

FIGURE 41.15 The wave function in the classically forbidden region.



EXAMPLE 41.7 Penetration distance of an electron

An electron is confined in a 2.0-nm-wide region with a potential-energy depth of 1.00 eV. What are the penetration distances into the classically forbidden region for an electron in the $n = 1$ and $n = 4$ states?

MODEL The electron is in the finite potential well whose energies and wave functions were shown in Figure 41.14a.

SOLVE The ground state has $U_0 - E_1 = 1.000 \text{ eV} - 0.068 \text{ eV} = 0.932 \text{ eV}$. Similarly, $U_0 - E_4 = 0.051$ eV in the $n = 4$ state. We can use Equation 41.41 to calculate

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \begin{cases} 0.20 \text{ nm} & n = 1 \\ 0.86 \text{ nm} & n = 4 \end{cases}$$

ASSESS These values are consistent with Figure 41.14a.

Quantum-Well Devices

In Part VI we developed a model of electrical conductivity in which the valence electrons of a metal form a loosely bound "sea of electrons." The typical speed of an electron is the rms speed:

$$v_{\mathrm{rms}} = \sqrt{\frac{3k_{\mathrm{B}}T}{m}}$$

where k_{B} is Boltzmann's constant. Hence at room temperature, where $v_{\mathrm{rms}} \approx 1 \times 10^5$ m/s, the de Broglie wavelength of a typical conduction electron is

$$\lambda \approx \frac{h}{mv_{\mathrm{rms}}} \approx 6 \text{ nm}$$

There is a range of wavelengths because the electrons have a range of speeds, but this is a typical value.

You've now seen many times that wave effects are significant only when the sizes of physical structures are comparable to or smaller than the wavelength. This is why the interference and diffraction of light are hard to observe and why the wave-like nature of matter becomes important only on microscopic scales. Because the de Broglie wavelength of conduction electrons is only a few nm, quantum effects are insignificant in electronic devices whose features are larger than about 100 nm. The electrons in macroscopic devices can be treated as classical particles, which is how we analyzed electric current in Chapter 31.

However, devices smaller than about 100 nm do exhibit quantum effects. Some semiconductor devices, such as the semiconductor lasers used in fiber-optic communications, now incorporate features only a few nm in size. Quantum effects play an important role in these devices.

FIGURE 41.16 A semiconductor diode laser with a single quantum well.

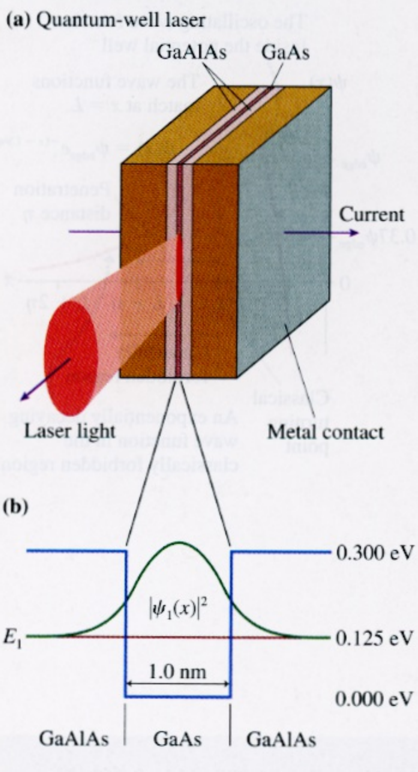


FIGURE 41.16a shows the construction of a *semiconductor diode laser.* Although the operating principles of diodes are beyond the scope of this textbook, we can note that a current travels through this device from left to right. In the center is a very thin layer of the semiconductor gallium arsenide (GaAs). It is surrounded on either side by layers of gallium aluminum arsenide (GaAlAs), and these in turn are embedded within the larger structure of the diode. The electrons within the central GaAs layer begin to emit laser light when the current through the diode exceeds some *threshold current.*

You can learn in a solid-state physics or materials engineering course that the electric potential energy of an electron is slightly lower in GaAs than in GaAlAs. This makes the GaAs layer a potential well for electrons, with higher-potential-energy GaAlAs "walls" on either side. As a result, the electrons become trapped within the thin GaAs layer. Such a device is called a **quantum-well laser.**

As an example, **FIGURE 41.16b** shows a quantum-well device with a 1.0-nm-thick GaAs layer in which the electron's potential energy is 0.300 eV lower than in the surrounding GaAlAs layers. A numerical solution of the Schrödinger equation finds that this potential well has only a *single* quantum state, $n = 1$ with $E_1 = 0.125$ eV. Every electron trapped in this quantum well has the *same* energy—a very nonclassical result! The fact that the electron energies are so well defined, in contrast to the range of electron energies in bulk material, is what makes this a useful device. You can also see from the probability density $|\psi|^2$ that the electrons are more likely to be found in the center of the layer than at the edges. This concentration of electrons makes it easier for the device to begin laser action.

Nuclear Physics

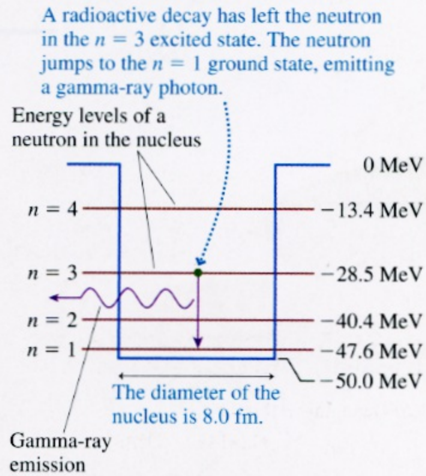
The nucleus of an atom consists of an incredibly dense assembly of protons and neutrons. The positively charged protons exert extremely strong electric repulsive forces on each other, so you might wonder how the nucleus keeps from exploding. During the 1930s, physicists found that protons and neutrons also exert an *attractive* force on each other. This force, one of the fundamental forces of nature, is called the *strong force.* It is the force that holds the nucleus together.

The primary characteristic of the strong force, other than its strength, is that it is a *short-range* force. The attractive strong force between two *nucleons* (a nucleon is either a proton or a neutron; the strong force does not distinguish between them) rapidly decreases to zero if they are separated by more than about 2 fm. This is in sharp contrast to the long-range nature of the electric force.

A reasonable model of the nucleus is to think of the protons and neutrons as particles in a nuclear potential well that is created by the strong force. The diameter of the potential well is equal to the diameter of the nucleus (this varies with atomic mass), and nuclear physics experiments have found that the depth of the potential well is $\approx$50 MeV.

The real potential well is three-dimensional, but let's make a simplified model of the nucleus as a one-dimensional potential well. **FIGURE 41.17** shows the potential energy of a neutron along an x-axis passing through the center of the nucleus. Notice that the zero of energy has been chosen such that a "free" neutron, one outside the nucleus, has $E = 0$. Thus the potential energy inside the nucleus is -50 MeV. The 8.0 fm diameter shown is appropriate for a nucleus having atomic mass number $A \approx 40$, such as argon or potassium. Lighter nuclei will be a little smaller, heavier nuclei somewhat larger. (The potential-energy diagram for a proton is similar, but is complicated a bit by the electric potential energy.)

FIGURE 41.17 There are four allowed energy levels for a neutron in this nuclear potential well.



A numerical solution of the Schrödinger equation finds the four stationary states shown in Figure 41.17. The wave functions have been omitted, but they look essentially identical to the wave functions in Figure 41.14a. The major point to note is that the allowed energies differ by several *million* electron volts! These are enormous energies compared to those of an electron in an atom or a semiconductor. But recall that the energies of a particle in a rigid box, $E_n = n^2h^2/8mL^2$, are proportional to $1/L^2$. Our previous examples, with nanometer-size boxes, found energies in the eV range. When the box size is reduced to femtometers, the energies jump up into the MeV range.

It often happens that the nuclear decay of a radioactive atom leaves a neutron in an excited state. For example, Figure 41.17 shows a neutron that has been left in the $n = 3$ state by a previous radioactive decay. This neutron can now undergo a quantum jump to the $n = 1$ ground state by emitting a photon with energy

$$E_{\text{photon}} = E_3 - E_1 = 19.1 \text{ MeV}$$

and wavelength

$$\lambda_{\text{photon}} = \frac{c}{f} = \frac{hc}{E_{\text{photon}}} = 6.50 \times 10^{-5} \text{ nm}$$

This photon is $\approx 10^7$ times more energetic, and its wavelength $\approx 10^7$ times smaller, than the photons of visible light! These extremely high-energy photons are called **gamma rays.** Gamma-ray emission is, indeed, one of the primary processes in the decay of radioactive elements.

Our one-dimensional model cannot be expected to give accurate results for the energy levels or gamma-ray energies of any specific nucleus. Nonetheless, this model does provide a reasonable understanding of the energy-level structure in nuclei and correctly predicts that nuclei can emit photons having energies of several million electron volts. This model, when extended to three dimensions, becomes the basis for the *shell model* of the nucleus in which the protons and neutrons are grouped in various shells analogous to the electron shells around an atom that you remember from chemistry. You can learn more about nuclear physics and the shell model in Chapter 43.

41.7 Wave-Function Shapes

Bound-state wave functions are standing de Broglie waves. In addition to boundary conditions, two other factors govern the shapes of wave functions:

1. The de Broglie wavelength is inversely dependent on the particle's speed. Consequently, the node spacing is smaller (shorter wavelength) where the kinetic