Brief Math Interlude

Suppose that we have $f\left(x,y,z\right)=0$

Then x can be imagined as a function of y & z:

$$dx=\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz$$

Also, y can be imagined as a function of x & z:

$$dy=\left(\frac{\partial y}{\partial x}\right)_z dx+\left(\frac{\partial y}{\partial z}\right)_x dz$$

If we substitute the second equation into the first one:

$$dx=\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial x}\right)_z dx+\left[\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x+\left(\frac{\partial x}{\partial z}\right)_y\right]dz$$

Now, only 2 of the 3 coordinates are independent. Let's choose x and z as being independent. Then the previous equation must be true for all sets of dx and dz.

a) if we choose the set $(dx \neq 0, dz = 0)$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$$

b) if we choose the set $(dx = 0, dz \neq 0)$

$$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -\left(\frac{\partial x}{\partial z}\right)_y$$

$$\text{or } \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$$

Returning to our particular situation, we have 3 "coordinates", (ϕ, x, t), with x and t being independent.

We'll use result b)

$$\left(\frac{\partial \phi}{\partial x}\right)_t \left(\frac{\partial x}{\partial t}\right)_\phi \left(\frac{\partial t}{\partial \phi}\right)_x = -1$$

$$\text{or} \left(\frac{\partial x}{\partial t}\right)_\phi = \frac{-\left(\partial \phi / \partial t\right)_x}{\left(\partial \phi / \partial x\right)_t} \qquad (\text{with } \phi = kx - \omega t)$$

Let's look at each term separately:

1) $\left(\frac{\partial x}{\partial t}\right)_{\phi}$

constant ϕ means a point on the wave profile that always has the same displacement, y.

$\frac{\partial x}{\partial t}$ is a velocity, representing how fast the point moves with the profile.

$\Rightarrow$ phase velocity, v_p (velocity of point of constant phase)

2) $$\left(\frac{\partial\phi}{\partial t}\right)_x = \frac{\partial\left(kx-\omega t\right)}{\partial t} = -\omega$$

3) $$\left(\frac{\partial\phi}{\partial x}\right)_t = \frac{\partial\left(kx-\omega t\right)}{\partial x} = k$$

so, $$\left(\frac{\partial x}{\partial t}\right)_\phi = \frac{-\left(\partial\phi/\partial t\right)_x}{\left(\partial\phi/\partial x\right)_t}$$ becomes $$v_p = \frac{\omega}{k}$$