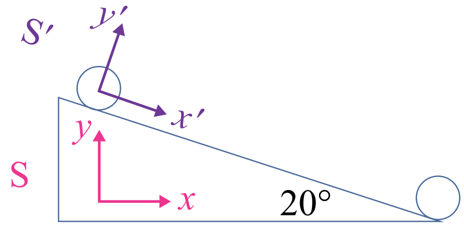
Invariant Quantities Part I: Cartesian Coordinates



In Phys 211, we found that when solving problems in mechanics, we got to choose the coordinate system, and some just make the problem easier than others. In the image to the left, Cher defines her coordinate system (S) with the x-axis aligned horizontally on the page. According to Cher, the ball starts at position $(x_1, y_1) = (0, h)$ and ends at position $(x_2, y_2) = (x_2, 0)$. Dionne defines her coordinate system (S′, shown on top) aligned with the ramp, such that the ball starts at position $(x'_1, y'_1) = (0,0)$ and ends at $(x'_2, y'_2) = (x'_2, 0)$.

1. According to Cher, what is Δx and Δy of the ball?

2. According to Dionne, what is $\Delta x'$ and $\Delta y'$ of the ball?

3. Both Cher and Dionne should agree on the total distance (*d)* the ball has traveled.
    a. What is Cher's expression for *d* in her coordinate system?

    b. What is Dionne's expression for *d* in her coordinate system?

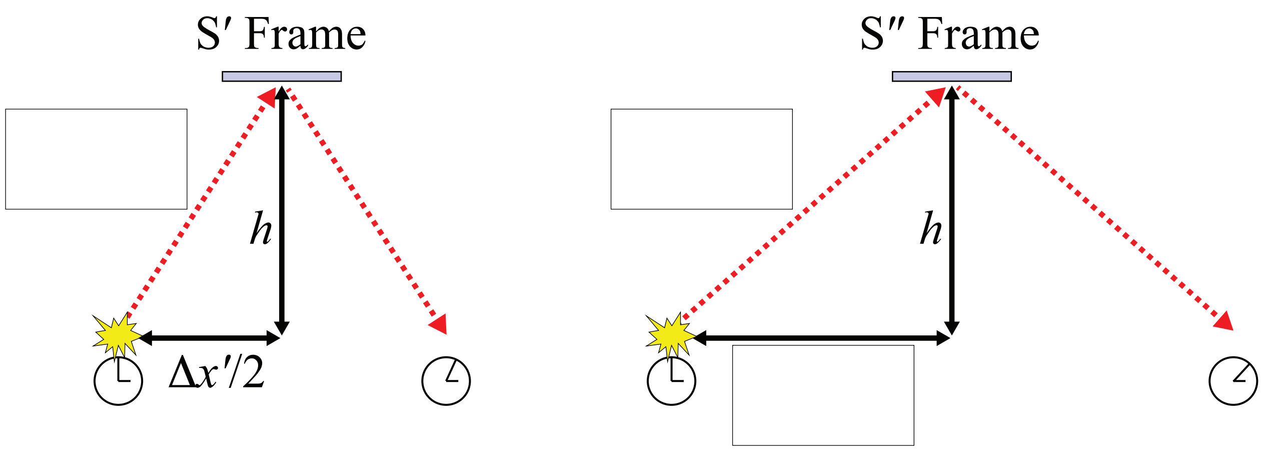
    c. Set the two expressions equal to one another. Do you believe that this equality is true? Why?

    d. Your equation above is an expression of what mathematical theorem?

In Cartesian coordinates, we call the total distance between two points *invariant* in geometry because it has the same value in any Cartesian coordinate system.

Invariant Quantities Part II: Revisiting the Time Clock

In our time clock, we also had a quantity that does not vary; the clock's height, h. Let's use that fact to find another invariant quantity in special relativity. Previously, our example usage of the time clock illustrated a "special case" where the proper time is measured in one frame. We want to relate distance and time between two frames in the most general way possible. Imagine we have two time clocks, and *both* are moving relative to some frame S. Let's call these two frames S′ ("S prime") and S″ ("S double prime"). In S′, the distance and time between the two events (emission of light and receiving bounced signal) is $\Delta x'$ and $\Delta t'$. In Frame S″, these are $\Delta x''$ and $\Delta t''$.



1. Fill in the boxes in the diagram above to express the distances of each triangle leg in terms of c, $\Delta x'$, $\Delta t'$, $\Delta x''$, and $\Delta t''$.
2. Use the Pythagorean theorem to write an equation relating the three measured distances in the S′ frame.

3. Do the same for the S″ frame.

4. Combine your previous two equations in order to eliminate h. Simplify and group all S′ terms on one side, and all S″ terms on the other side.

You've found a way to combine quantities in each reference frame to produce a quantity that all observers can agree upon; their calculation of this quantity will be the same regardless of their reference frame. This is called the **spacetime interval**, *s*, defined such that $s^2 = (c\Delta t)^2 - (\Delta x)^2$.

5. What is the spacetime interval between two events measured by the same clock? Is s^2 greater than, less, than, or equal to zero? These are called "time-like" events; there exists a frame where two events happen in the same place.

6. What is the spacetime interval between two simultaneous events? Is s^2 greater than, less, than, or equal to zero? This is called a "space-like" separation; there exists a frame where two events are simultaneous.

7. What is the significance of $s^2 = 0$?

We'll come back to these a bit more when we introduce spacetime diagrams and causality.