The Effect of Spring Mass on the Oscillation Frequency

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The purpose of this note is to calculate the effect of the spring mass on the oscillation frequency of an object hanging at the end of a spring. The goal is to find the limitations to a frequently-quoted rule that 1/3 the mass of the spring should be added to to the mass of the hanging object. This calculation was prompted by a student laboratory exercise in which it is normally seen that the frequency is somewhat lower than this rule would predict.

Consider a mass M hanging from a spring of unstretched length l, spring constant k, and mass m. If the mass of the spring is neglected, the oscillation frequency would be $\omega = \sqrt{k/M}$. The quoted rule suggests that the effect of the spring mass would be to replace M by $M + m/3$ in the equation for ω. This result can be found in some introductory physics textbooks, including, for example, Sears, Zemansky and Young, *University Physics*, 5th edition, sec. 11-5. The derivation assumes that all points along the spring are displaced linearly from their equilibrium position as the spring oscillates. This note will examine more general cases for the masses, including the limit $M = 0$. An appendix notes how the linear oscillation assumption breaks down when the spring mass becomes large.

Let the positions along the unstretched spring be labeled by x, running from 0 to L, with 0 at the top of the spring, and L at the bottom, where the mass M is hanging. When the spring is stretched, label the corresponding positions by a function $y(x)$. The motion of the spring will be described by the time variation of y. The length of the stretched spring will be denoted $Y = y(L)$. The top of the spring is fixed, at $y(0) = 0$. It will be assumed that the spring oscillations are not so large that they cause the spring coils to bump into each other, so the motion is just determined by the elastic extension and compression of the spring.

We begin with the derivation of the wave equation for the spring, which is a well-known result, and then incorporate the boundary condition describing the motion of the hanging mass.

Consider a tiny element dx of the spring, which stretches to a length dy. Scaling the spring constant down from the whole spring to just the element dx gives the spring constant for just the length dx to be kL/dx. (The spring constant of a spring is inversely proportional to its length.) The change in length of this element is $dy - dx$. Therefore the tension on this element is

$$T(x) = \frac{kL}{dx}(dy - dx) = kL\left(\frac{dy}{dx} - 1\right). \tag{1}$$

Note that if the stretching were linear, then $dy/dx = Y/L + 1$, so that the tension $T(x)$ would reduce to $k(Y - L)$, as expected.

The element dx has mass mdx/L, and is acted on by its weight, plus the difference in tension on the two ends, giving a total force

$$dF(x) = \frac{mg}{L}dx + (T(x + dx) - T(x)). \tag{2}$$

The acceleration is $\ddot{y}$, so Newton's law for the element may be written using (1) as

$$dF(x) = \frac{mg}{L}dx + kL\frac{d}{dx}\left(\frac{dy}{dx} - 1\right)dx = \left(\frac{m}{L}dx\right)\ddot{y}. \tag{3}$$

This may be written as

$$\ddot{y} - \frac{kL^2}{m}y'' = g. \tag{4}$$

Equation (4) applies to any point on the spring except $x = L$, where the additional mass M is hanging. There, the force is

$$F(L) = Mg - T(L) = Mg - kL(y'(L) - 1). \tag{5}$$

Newton's Law at this point gives the boundary condition for $Y = y(L)$,

$$M\ddot{Y} = F(L) = Mg - kL(y'(L) - 1). \tag{6}$$

The general motion of the spring is obtained by solving (4) with boundary condition (6) at $x = L$ and boundary condition $y(0) = 0$ at the stationary point $x = 0$. It is convenient to subtract the static equilibrium solution $y_0(x)$, and work instead with the displacement $u(x,t) = y(x,t) - y_0(x)$ from equilibrium. The static solution is readily found to be

$$y_0(x) = x\left[1 + \frac{g}{kL}(M + m)\right] - \frac{mgx^2}{2kL^2}. \tag{7}$$

Then the equation of motion (4) becomes

$$\ddot{u} - \frac{kL^2}{m}u'' = 0, \tag{8}$$

and the boundary conditions (6) imply

$$u(0,t) = 0, \qquad \ddot{u}(L) + \frac{kL}{M}u'(L) = 0 \tag{9}$$

Equation (8) is recognized as the wave equation for the spring, describing longitudinal waves with velocity $v_s = L\sqrt{k/m}$. An equivalent but simpler expression for the boundary conditions (9) can be obtained using the spring equation of motion (8), which gives

$$u(0,t) = 0, \qquad u''(L,t) + \frac{m}{ML}\,u'(L,t) = 0. \tag{10}$$

We will consider the special case where the entire spring vibrates in phase at a common frequency ω, so that the motion can be expressed as

$$u(x,t) = u(x)\cos(\omega t) \tag{11}$$

with ω to be determined. The equation of motion then reduces to

$$u''(x) = -\frac{m\omega^2}{kL^2}u(x) = -\left(\frac{\omega}{v_s}\right)^2 u(x) \tag{12}$$

with boundary conditions

$$u(0) = 0, \qquad u'(L) = \frac{M\omega^2}{kL}u(L). \tag{13}$$

The solution to (12) with $u(0) = 0$ must take the form

$$u(x) = D\sin(\alpha x), \tag{14}$$

with amplitude D and $\alpha = (m/k)^{1/2}\omega/L = \omega/v_s$. Substituting this solution into (13) gives, at $x = L$,

$$\tan(\alpha L) = \frac{\alpha k L}{M\omega^2} = \frac{\rho}{\alpha L} \tag{15}$$

where $\rho = m/M$ is the ratio of the spring mass to the hanging mass. If $\omega_0 = \sqrt{k/M}$ is the angular frequency when the spring is massless, (15) can also be expressed as

$$\tan\left(\sqrt{\rho}\frac{\omega}{\omega_0}\right) = \sqrt{\rho}\frac{\omega_0}{\omega}. \tag{16}$$

The solutions in the limit $\rho = 0$ have a somewhat different form, since $\alpha = 0$ in that case. When $\rho = 0$, the displacement is simply linear: $u(x) = Dx/L$, and $\omega = \omega_0$. In the opposite limit, $\rho \to \infty$, we can set $M = 0$ in the original equations and find $\cos(\alpha L) = 0$, which implies $\omega = n\pi v_s/2$ for any odd n. In this case, there is no hanging mass, and the solutions are longitudinal waves on the spring with a fixed boundary on one end and free boundary on the other. The fundamental angular frequency is $\omega = \pi v_s/2L$, as usual for longitudinal waves with these boundary conditions.

For intermediate values of ρ, there will be infinitely many solutions of the form $u_n(x) = A\sin(\alpha_n x)$. The equations of motion are linear, so the general motion will be a linear combination of modes, with the higher oscillations becoming more important as the ratio ρ becomes larger. See the appendix for a more complete discussion of the motion of the spring. In the remainder of this note, we will take the oscillation frequency of the spring to be the fundamental frequency, the one which has a smooth limit to ω_0 when the mass of the spring goes to zero.

To solve (16) we can use the fact that

$$\frac{z}{\tan(z)} = 1 - \frac{z^2}{3} - \delta(z^2) \tag{17}$$

where the function $\delta(z)$ can be expanded in terms of Bernoulli numbers,

$$\delta(z) = \sum_{k=2}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} z^k = \frac{z^2}{45} + \frac{2z^3}{945} + \frac{z^4}{4725} + \dots \tag{18}$$

or in terms of the Riemann zeta function,

$$\delta(z) = \frac{2z^2}{\pi^4} \sum_{k=0}^{\infty} \zeta(2k+4) \left(\frac{z}{\pi^2}\right)^k . \tag{19}$$

Equation (16) may then be rewritten as

$$\frac{\omega^2}{\omega_0^2} = \frac{1 - \delta(\rho\omega^2/\omega_0^2)}{1 + \rho/3} \tag{20}$$

It is convenient to express the solution ω in terms of an effective mass $M' = M + cm$ where c is a fraction of the spring mass which must be added to the hanging mass to obtain the correct oscillation frequency, $\omega = \sqrt{k/M'}$. The expression (20) may be rewritten in terms of the fraction c as

$$\frac{\omega^2}{\omega_0^2} = \left(\frac{k}{M + cm}\right)\left(\frac{M}{k}\right) = \frac{M}{M + cm} = \frac{1}{1 + c\rho}. \tag{21}$$

Then the fraction of the spring mass which must be added to the hanging mass is

$$c = \frac{1}{1-\delta}\left(\frac{1}{3} + \frac{\delta}{\rho}\right) \quad \text{with} \quad \delta = \delta\left(\frac{\rho}{1 + c\rho}\right). \tag{22}$$

The general dependence of c on the mass ratio ρ is shown in Figure 1. For small ρ, the δ term can be dropped in (22), giving $c = 1/3$. This is the result derived in text books, including Sears, Zemansky and Young, *University Physics*, 5th ed., by much simpler means. Corrections can be obtained iteratively, beginning with $c = 1/3$ when evaluating δ on the right-hand side of (22), and then calculating improved values until the result converges.

In the opposite limit, when the hanging mass is removed entirely, and $\rho \to \infty$, c approaches a limiting value of $4/\pi^2 \approx 0.405$. (This limit corresponds to $z = c^{-1/2} = \pi/2$ in (17).) This is consistent with the result obtained by setting $M = 0$ from the beginning and using the fundamental frequency $\omega = \pi v_s/2$ found earlier, which implies

$$\omega(\rho \to \infty) = \frac{\pi}{2}\sqrt{\frac{k}{m}} = \sqrt{\frac{k}{M + cm}} \tag{23}$$

with $M = 0$ and $c = 4/\pi^2$.

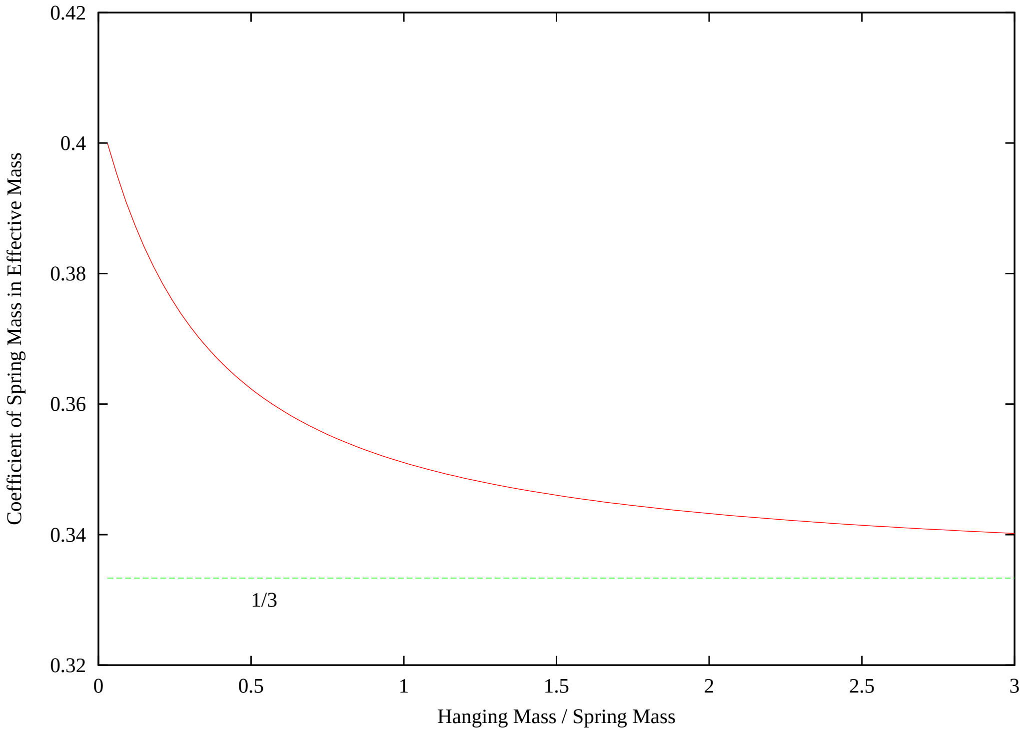

Figure 1: Dependence of the Effective Mass Parameter on the Ratio of Masses

Appendix: Motion of the Spring

When $\rho > 0$, it is possible to form an orthonormal set of functions describing the oscillations of the mass on the spring. Let $u_n(x) = \sqrt{2/L}\sin(\alpha_n x)$, where $\alpha_n = \omega_n/v_s$ is the n^{th} solution to equation (15). The boundary condition (13) can be used to show that the functions $u_n(x)$ are orthonormal with respect to an inner-product $(u|v) = \int_0^L u(x)v(x)dx + Lu(L)v(L)/\rho$, so that $(u_m|u_n) = \delta_{mn}$. A general solution can be constructed by adding a linear combination of the u_n to match the initial conditions. Suppose the hanging mass is simply pulled down a distance D from the equilibrium position and then released. When the spring is pulled down by applying a force $F = kD$, the position $y(x)$ of the spring is given by

$$y(x,0) = x\left[1 + \frac{g}{kL}(M+m) + \frac{F}{kL}\right] - \frac{mgx^2}{2kL^2} = y_0(x) + u(x) \qquad (24)$$

with equilibrium position $y_0(x)$ given by (7) and a linear displacement function $u(x) = Fx/kL = xD/L$. The displacement function u can be expanded as $u(x) = \sum_n b_n u_n(x)$ with coefficients

$$b_n = (u|u_n) = \frac{D}{L}\frac{u_n(L)}{\alpha_n^2} = \frac{D}{L}\frac{v_s^2}{\omega_n^2}\sqrt{\frac{2}{L}}\sin(\alpha_n L). \qquad (25)$$

The time-dependent displacement function is then

$$u(x,t) = \frac{2D}{L^2}\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\sin(\alpha_n L)\sin(\alpha_n x)\cos(\omega_n t). \qquad (26)$$

When ρ is small, only b_1 is significant and $u(x,t) \approx (Dx/L)\cos\omega_1 t$, but for larger ρ, the higher oscillation modes are excited. This is consistent with increasingly complicated composite motions seen in the laboratory when the hanging mass is reduced compared to the spring mass. The approximate treatment which finds $c = 1/3$ regardless of ρ assumes that the linear form $u(x) = Dx/L$ holds at all times, not just the initial instant. In fact, this is only true if $\rho \to 0$. In the opposite limit, where the hanging mass is absent, $\alpha_n = n\pi/2L$ with n odd, and $b_n = (4DL/\pi^2 n^2)\sqrt{2/L}$, which shows that the higher modes contribute significantly to the motion:

$$u(x,t) = \frac{8D}{\pi^2}\sum_{n\ \text{odd}}\frac{1}{n^2}\sin\left(\frac{n\pi x}{2L}\right)\cos\left(\frac{n\pi v_s t}{2L}\right). \qquad (27)$$