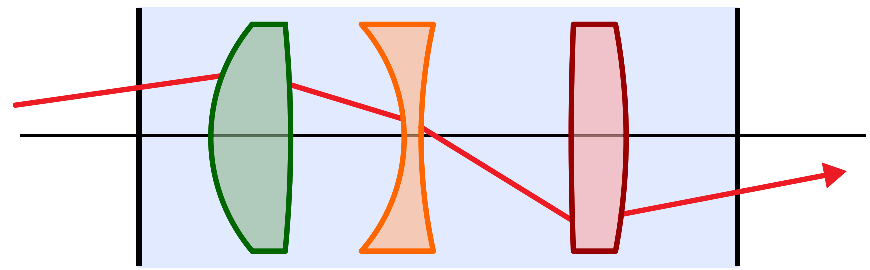
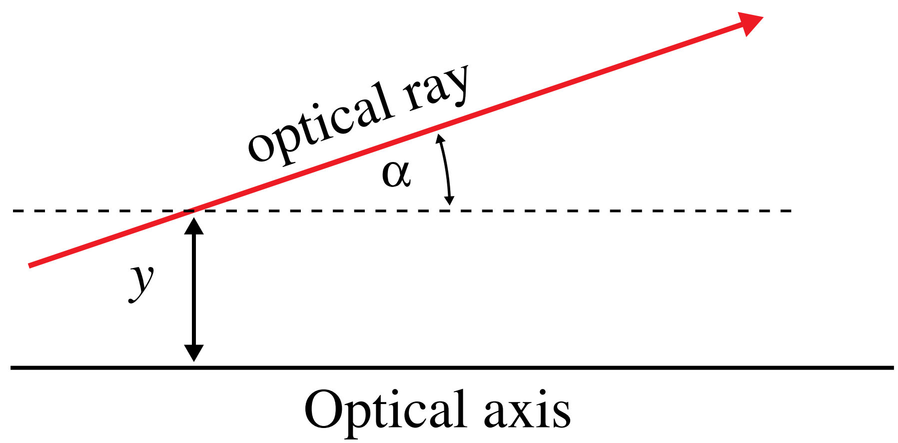
The Ray Vector



A light ray can be defined by two coordinates:

its "height", y

its slope, α



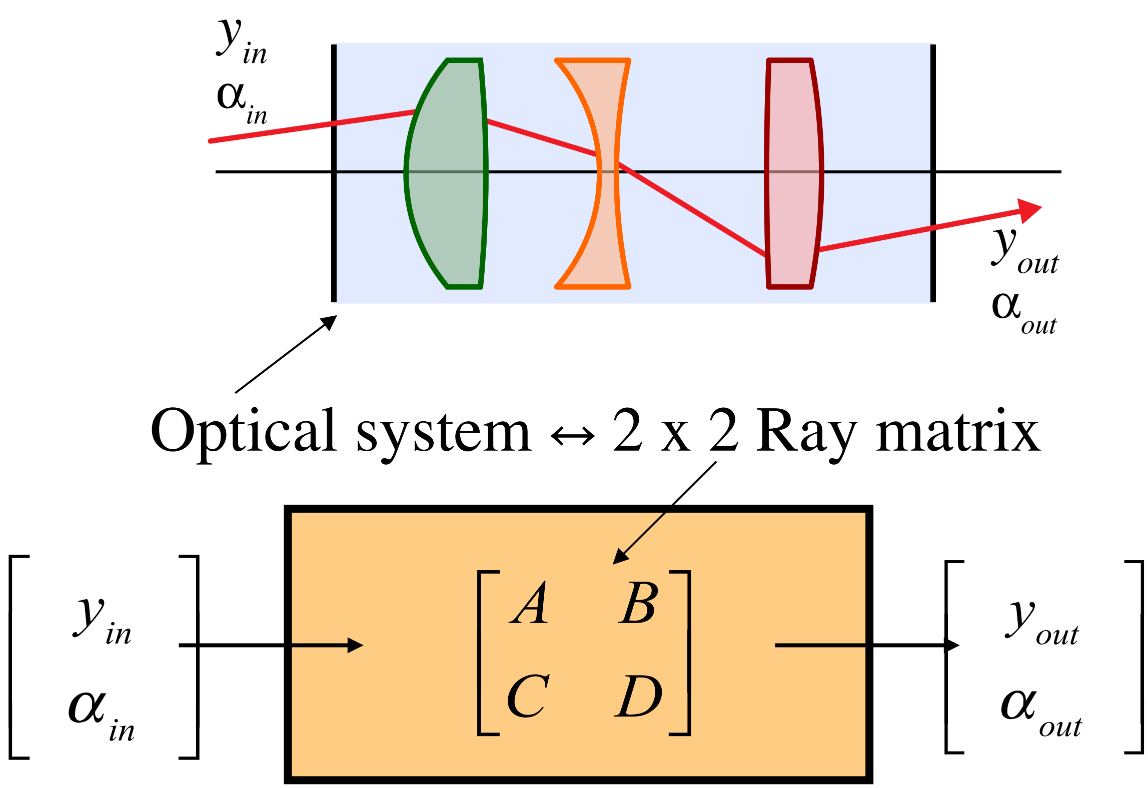
These parameters define a **ray vector**, which will change with distance, and/or as the ray propagates through optical interfaces and elements.

$$\begin{bmatrix} y \\ \alpha \end{bmatrix}$$

Ray Matrices

For many optical components, we can define 2 x 2 **ray matrices**.

An element’s affect on a ray is found by multiplying its ray vector.



Ray matrices can describe simple and complex systems.

These matrices are often called ABCD Matrices.

Ray matrices as derivatives

$$y_{out} = \frac{\partial y_{out}}{\partial y_{in}} y_{in} + \frac{\partial y_{out}}{\partial \alpha_{in}} \alpha_{in}$$

$$\alpha_{out} = \frac{\partial \alpha_{out}}{\partial y_{in}} y_{in} + \frac{\partial \alpha_{out}}{\partial \alpha_{in}} \alpha_{in}$$

Since the displacements and angles are assumed to be small, we can think in terms of partial derivatives.

spatial magnification $\frac{\partial y_{out}}{\partial y_{in}}$ (→ A)

$\frac{\partial y_{out}}{\partial \alpha_{in}}$ (→ B)

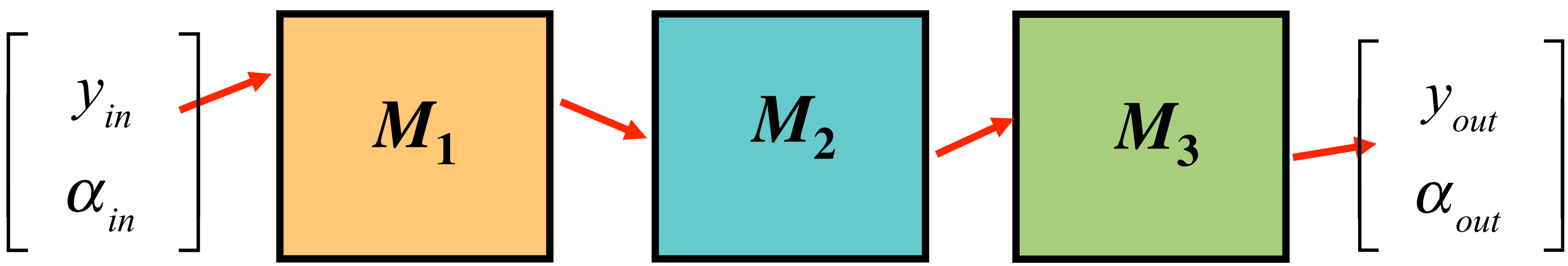
$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$\frac{\partial \alpha_{out}}{\partial y_{in}}$ (→ C)

$\frac{\partial \alpha_{out}}{\partial \alpha_{in}}$ **angular magnification** (→ D)

We can write these equations in matrix form.

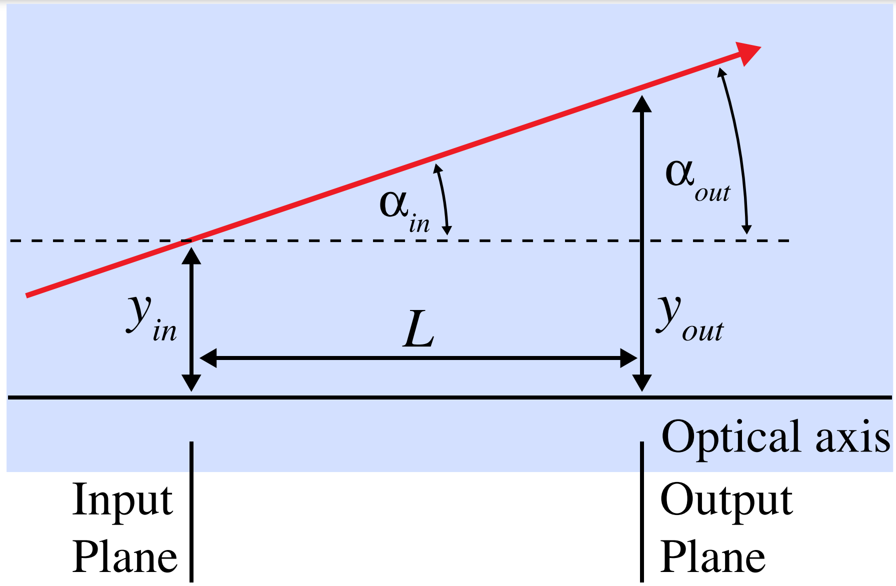
For cascaded elements, we simply multiply ray matrices.



$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = M_3 \left\{ M_2 \left(M_1 \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix} \right) \right\} = M_3 \, M_2 \, M_1 \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

Notice that the order looks opposite to what you think it should be, but it makes sense when you think about it.

Translation Matrix

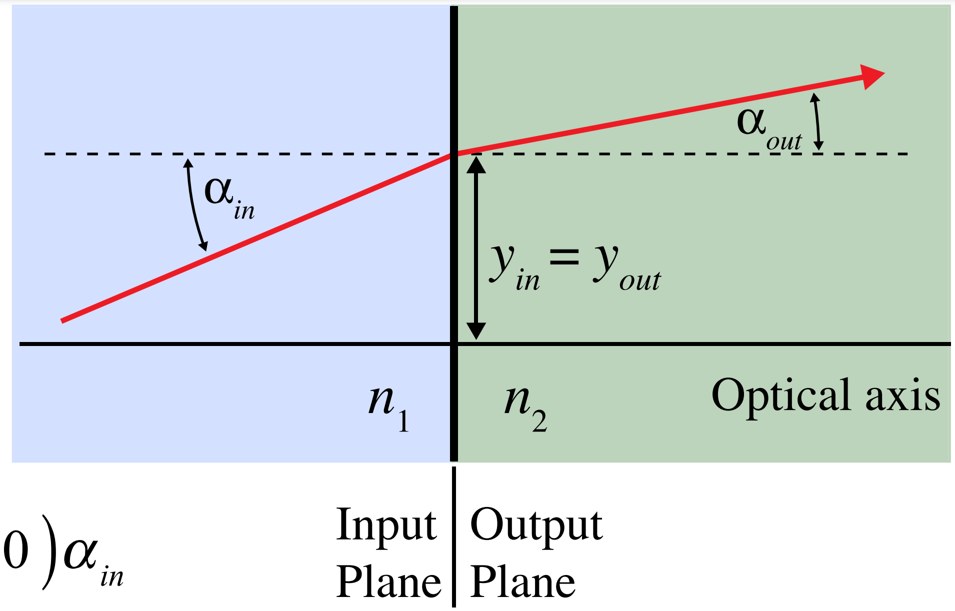


$$\alpha_{out} = \alpha_{in} \qquad y_{out} = y_{in} + L\tan\alpha_{in} \cong y_{in} + L\alpha_{in}$$

$$y_{out} = (1)y_{in} + (L)\alpha_{in}$$
$$\alpha_{out} = (0)y_{in} + (1)\alpha_{in}$$

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix} = \begin{bmatrix} 1 & x_{out} - x_{in} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

Refraction Matrix – Flat Interface

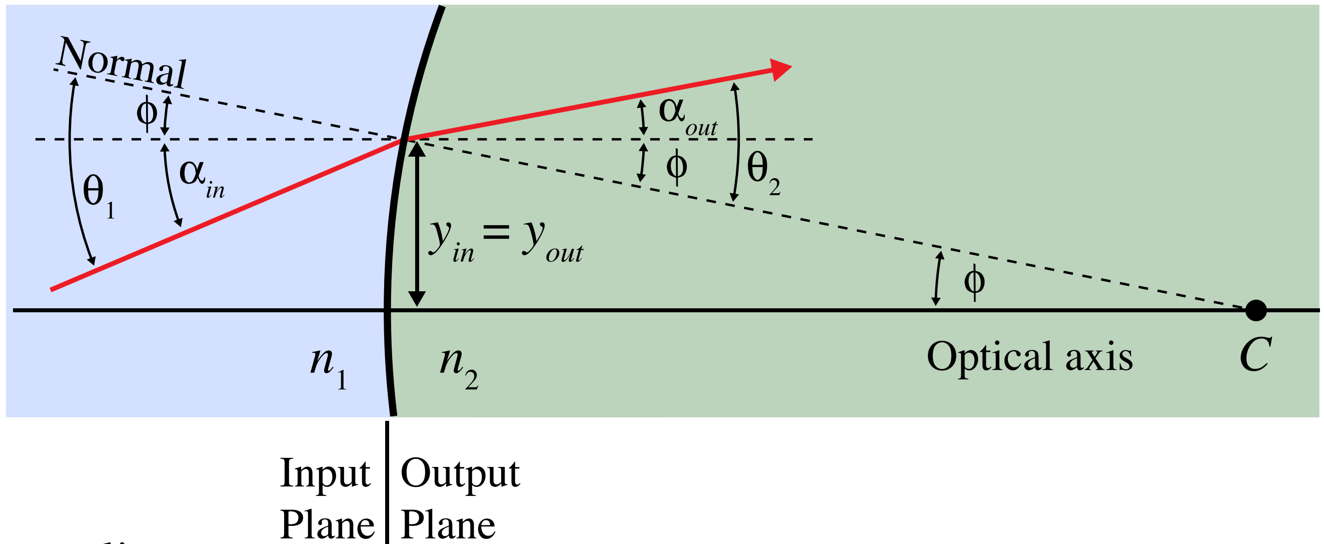


$$y_{out} = (1)\, y_{in} + (0)\, \alpha_{in}$$

Paraxial Snell's Law: $\quad n_1 \alpha_{in} = n_2 \alpha_{out} \quad \Rightarrow \quad \alpha_{out} = (0)\, y_{in} + \left(\frac{n_1}{n_2} \right) \alpha_{in}$

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{n_1}{n_2} \right) \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

Refraction Matrix – Curved Interface



$$\alpha_{out} = \theta_2 - \phi = \theta_2 - \frac{y_{in}}{R}$$

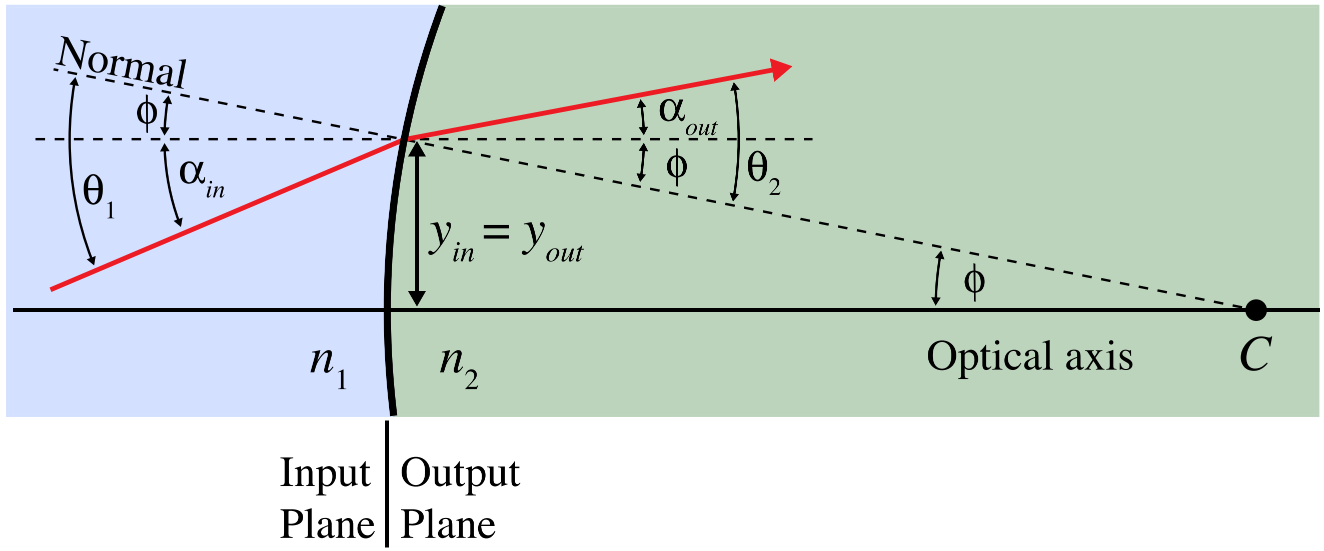
$$\alpha_{in} = \theta_1 - \phi = \theta_1 - \frac{y_{in}}{R}$$

$$\theta_1 = \alpha_{in} + \frac{y_{in}}{R} \qquad \theta_2 = \alpha_{out} + \frac{y_{in}}{R}$$

$$\textit{Paraxial Snell's Law}: \quad n_1\theta_1 = n_2\theta_2 \quad \rightarrow \quad n_1\left(\alpha_{in} + \frac{y_{in}}{R}\right) = n_2\left(\alpha_{out} + \frac{y_{in}}{R}\right)$$

$$\alpha_{out} = \left(\frac{n_1}{n_2}\right)\left(\alpha_{in} + \frac{y_{in}}{R}\right) - \frac{y_{in}}{R} = \boxed{\frac{1}{R}\left(\frac{n_1}{n_2} - 1\right)y_{in} + \left(\frac{n_1}{n_2}\right)\alpha_{in}}$$

Refraction Matrix – Curved Interface



$$y_{out} = (1)y_{in} + (0)\alpha_{in}$$

$$\alpha_{out} = \left[\frac{1}{R}\left(\frac{n_1}{n_2} - 1\right)\right] y_{in} + \left(\frac{n_1}{n_2}\right)\alpha_{in}$$

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \left(\frac{n_1}{n_2}\right) \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

Concave surface: $R < 0$

Convex surface: $R > 0$

Reflection Matrix

$$\alpha_{out} = \theta_1 + \theta_2 - \alpha_{in} \qquad \theta_1 = \alpha_{in} - \phi = \alpha_{in} - \frac{y_{in}}{-R} = \alpha_{in} + \frac{y_{in}}{R}$$

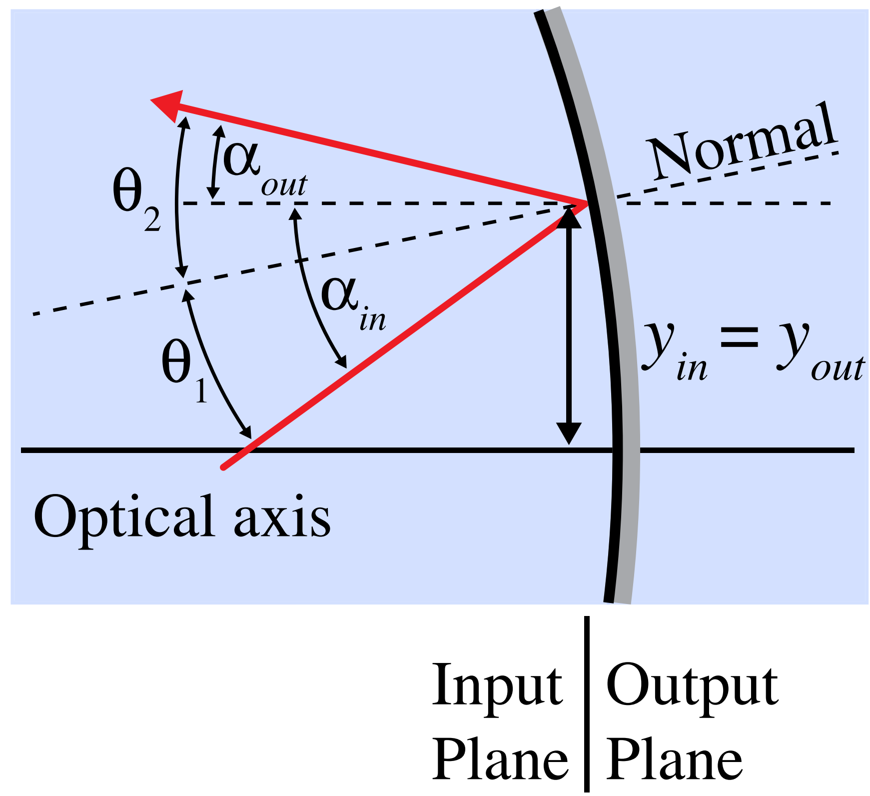
$$\textit{Law of Reflection}: \quad \theta_1 = \theta_2$$

$$\alpha_{out} = 2\theta_1 - \alpha_{in} = \alpha_{in} + \frac{2}{R} y_{in}$$

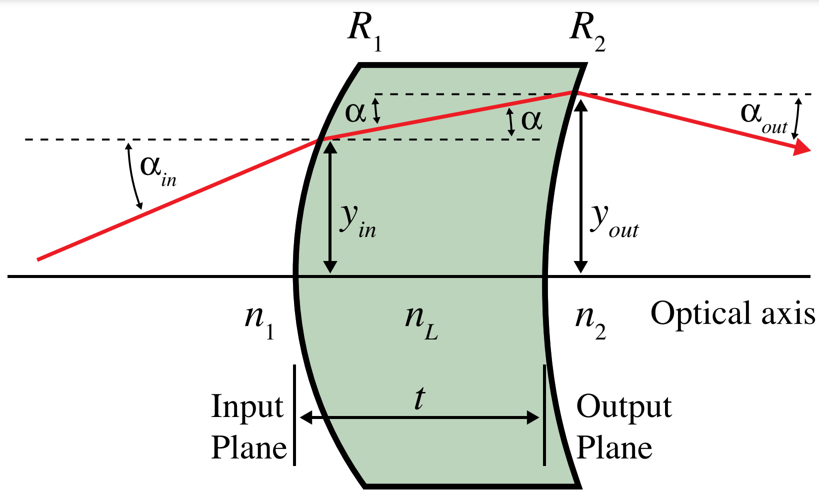
$$y_{out} = (1) y_{in} + (0) \alpha_{in}$$

$$\alpha_{out} = \left(\frac{2}{R}\right) y_{in} + (1) \alpha_{in}$$

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$



Thick Lens Matrix



Refraction at first surface:

$$\begin{bmatrix} y_1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_1}{n_L}-1\right) & \frac{n_1}{n_L} \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix} = M_1 \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

Translation from 1st surface to 2nd surface:

$$\begin{bmatrix} y_2 \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha \end{bmatrix} = M_2 \begin{bmatrix} y_1 \\ \alpha \end{bmatrix}$$

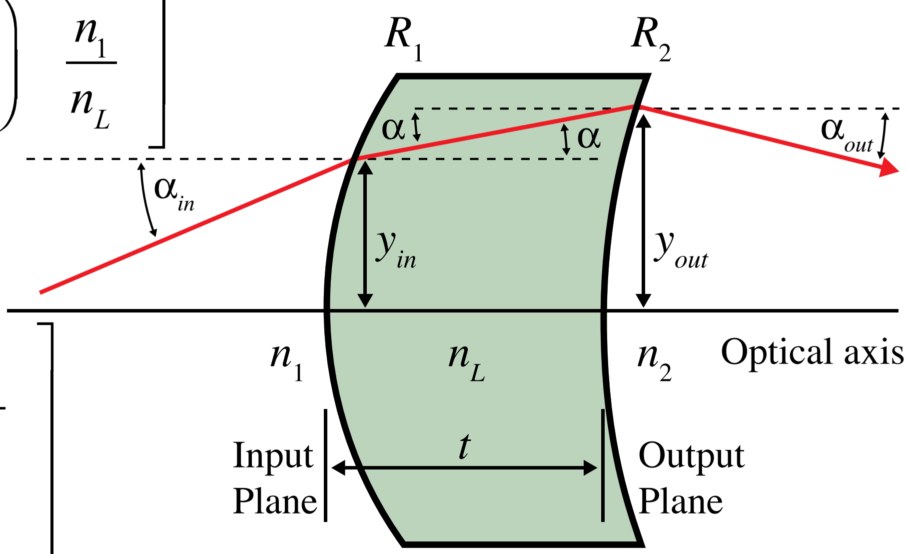
Refraction at second surface:

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n_2}-1\right) & \frac{n_L}{n_2} \end{bmatrix} \begin{bmatrix} y_2 \\ \alpha \end{bmatrix} = M_3 \begin{bmatrix} y_2 \\ \alpha \end{bmatrix}$$

Thick Lens Matrix

Thick lens matrix: $M = M_3 M_2 M_1$

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n_2}-1\right) & \frac{n_L}{n_2} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n_1}{n_L}-1\right) & \frac{n_1}{n_L} \end{bmatrix}$$

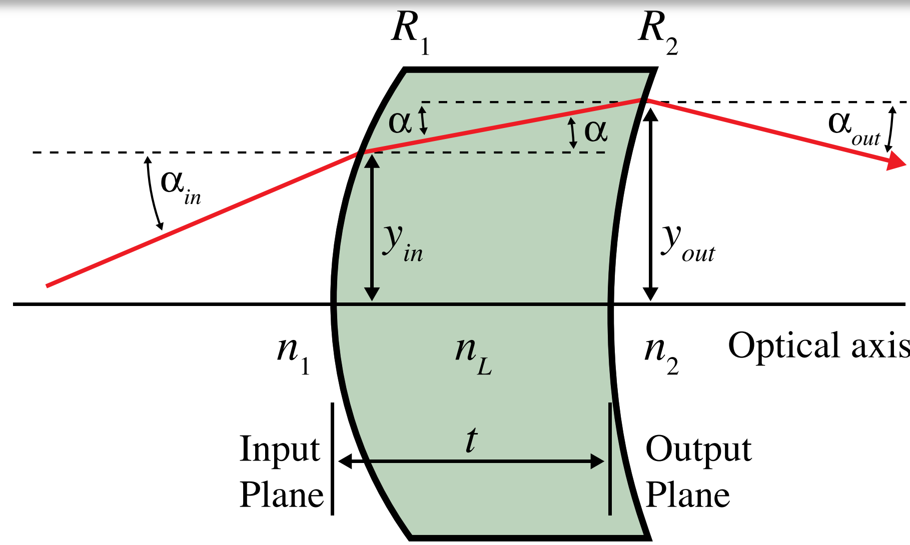


Assuming $n_1 = n_2 = n$:

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n_L}{n}-1\right) & \frac{n_L}{n} \end{bmatrix} \begin{bmatrix} 1+\frac{t}{R_1}\left(\frac{n}{n_L}-1\right) & t\frac{n}{n_L} \\ \frac{1}{R_1}\left(\frac{n}{n_L}-1\right) & \frac{n}{n_L} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\frac{t}{R_1}\left(\frac{n}{n_L}-1\right) & t\frac{n}{n_L} \\ \frac{1}{R_2}\left(\frac{n_L}{n}-1\right)\left[1+\frac{t}{R_1}\left(\frac{n}{n_L}-1\right)\right]-\frac{1}{R_1}\left(\frac{n_L}{n}-1\right) & 1-\frac{t}{R_2}\left(\frac{n}{n_L}-1\right) \end{bmatrix}$$

Thick Lens Matrix



$$M = \begin{bmatrix} 1 + \frac{t}{R_1}\left(\frac{n}{n_L} - 1\right) & t\frac{n}{n_L} \\ \frac{1}{R_2}\left(\frac{n_L}{n} - 1\right)\left[1 + \frac{t}{R_1}\left(\frac{n}{n_L} - 1\right)\right] - \frac{1}{R_1}\left(\frac{n_L}{n} - 1\right) & 1 - \frac{t}{R_2}\left(\frac{n}{n_L} - 1\right) \end{bmatrix}$$

Thin Lens Matrix

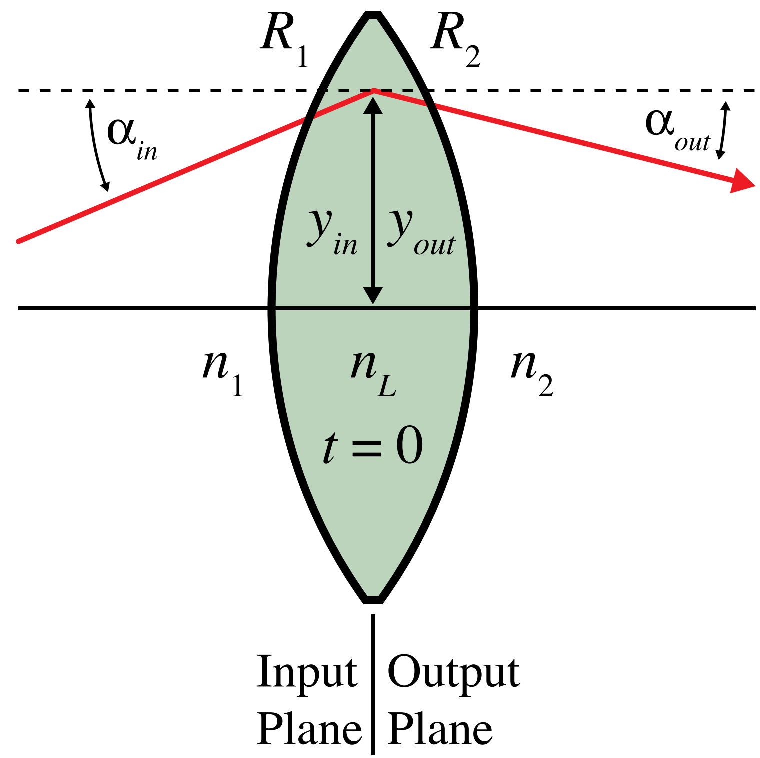
The thin lens matrix is found by setting $t = 0$:

$$\textit{Thin lens matrix}:$$

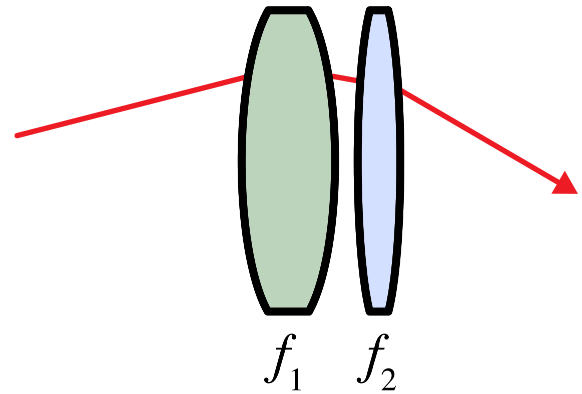
$$M = \begin{bmatrix} 1 & 0 \\ \left(\frac{n_L}{n} - 1\right)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}$$

$$\textit{but} \quad \frac{1}{f} = \left(\frac{n_L}{n} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$



Consecutive thin lenses



Suppose we have two lenses right next to each other (with no space in between).

$$M_{tot} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_1 - 1/f_2 & 1 \end{bmatrix}$$

$$1/f_{\text{tot}} = 1/f_1 + 1/f_2$$

So two consecutive lenses act as one whose focal length is computed by the resistive sum.

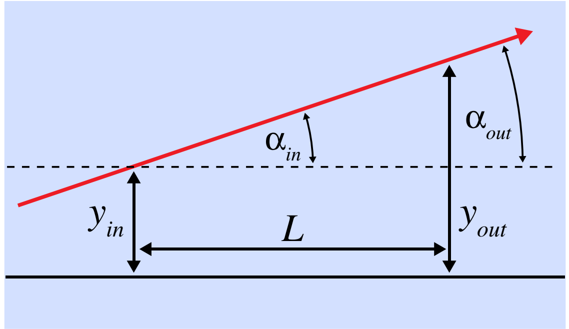
As a result, we define a measure of inverse lens focal length, the diopter.

1 diopter = 1 m^{-1}

Summary of Matrix Methods

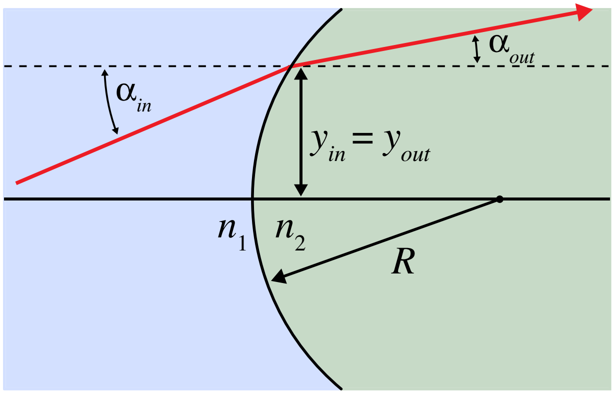
Translation Matrix:

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$



Refraction matrix,
spherical interface:

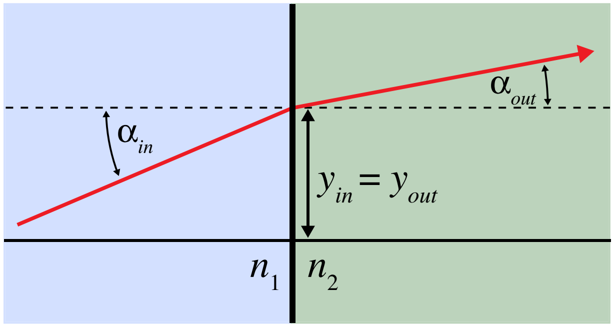
$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{bmatrix}$$



(+R): Convex (–R): Concave

Refraction matrix,
plane interface:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$

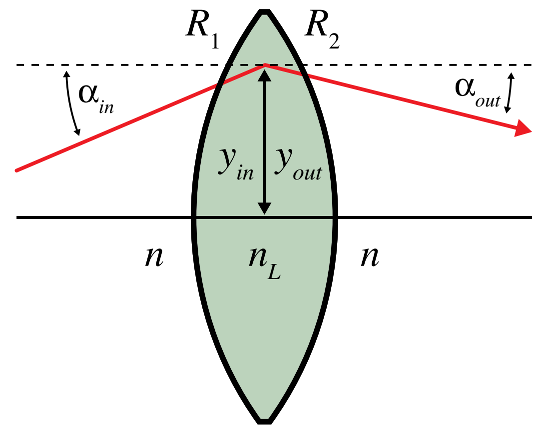


Summary of Matrix Methods

Thin-lens Matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

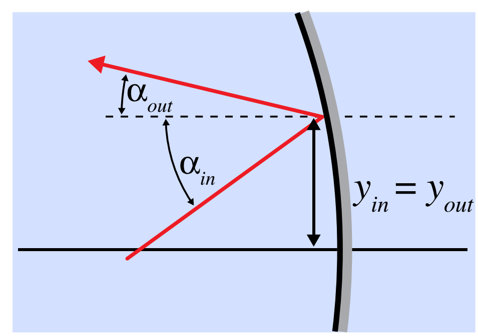
$$\frac{1}{f} = \left(\frac{n_L}{n} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$



(+f): Converging (–f): Diverging

Spherical mirror matrix:

$$M = \begin{bmatrix} 1 & 0 \\ \dfrac{2}{R} & 1 \end{bmatrix}$$



(+R): Convex (–R): Concave

System Ray-Transfer Matrix

Any paraxial optical system, no matter how complicated, can be represented by a 2x2 optical matrix. This matrix M is usually denoted

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

A useful property of this matrix is that

$$\text{Det}\, M = AD - BC = \frac{n_{in}}{n_{out}}$$

where n_{in} and n_{out} are the refractive indices of the input and output media of the optical system. Usually, the medium will be air on both sides of the optical system and

$$\text{Det}\, M = AD - BC = \frac{n_{in}}{n_{out}} = 1$$

System Ray-Transfer Matrix

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix} \quad \Rightarrow$$

$$y_{out} = Ay_{in} + B\alpha_{in}$$

$$\alpha_{out} = Cy_{in} + D\alpha_{in}$$

Let's examine the implications when any of the four elements of the system matrix is equal to zero.

System Ray-Transfer Matrix

Let's see what happens when $D = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in} + B\alpha_{in}$$

$$\alpha_{out} = Cy_{in}$$

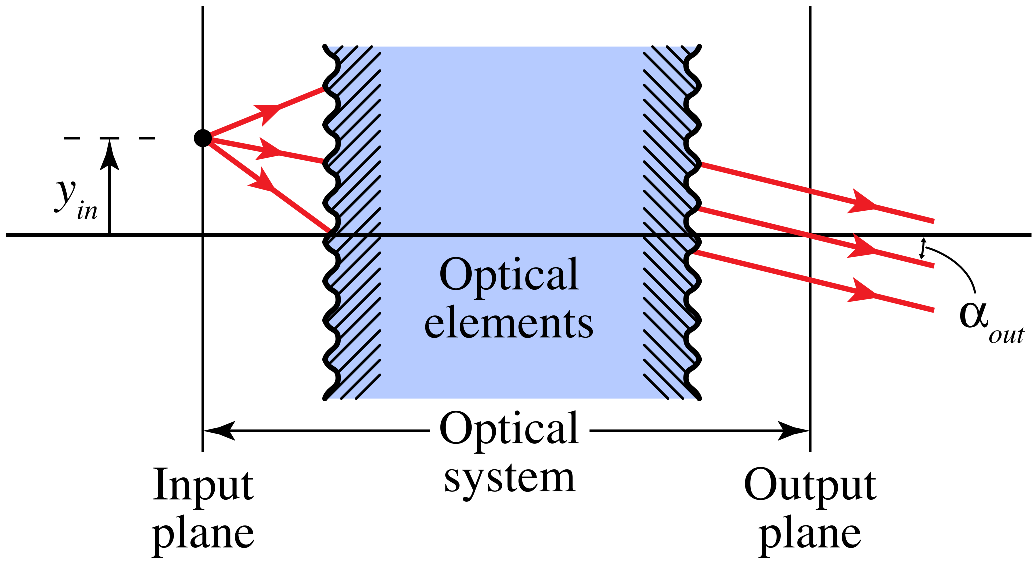
System Ray-Transfer Matrix

Let's see what happens when $D = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in} + B\alpha_{in}$$

$$\alpha_{out} = Cy_{in}$$



When $D = 0$, the input plane for the optical system is the input focal plane.

System Ray-Transfer Matrix

Let's see what happens when $A = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

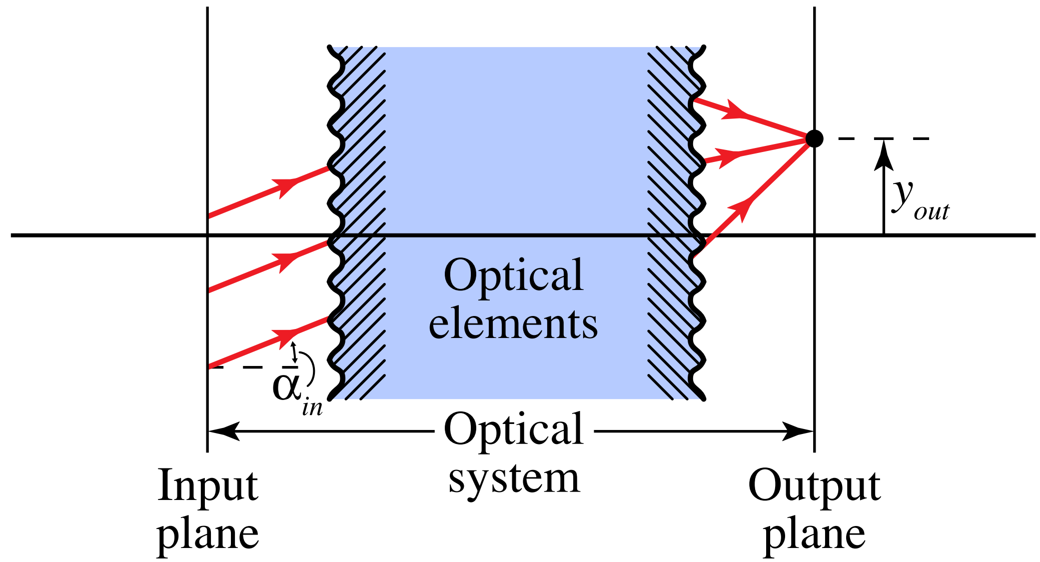
$$y_{out} = B\alpha_{in}$$

$$\alpha_{out} = Cy_{in} + D\alpha_{in}$$

System Ray-Transfer Matrix

Let's see what happens when $A = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$



$$y_{out} = B\alpha_{in}$$

$$\alpha_{out} = Cy_{in} + D\alpha_{in}$$

When $A = 0$, the output plane for the optical system is the output focal plane.

System Ray-Transfer Matrix

Let's see what happens when $C = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in} + B\alpha_{in}$$

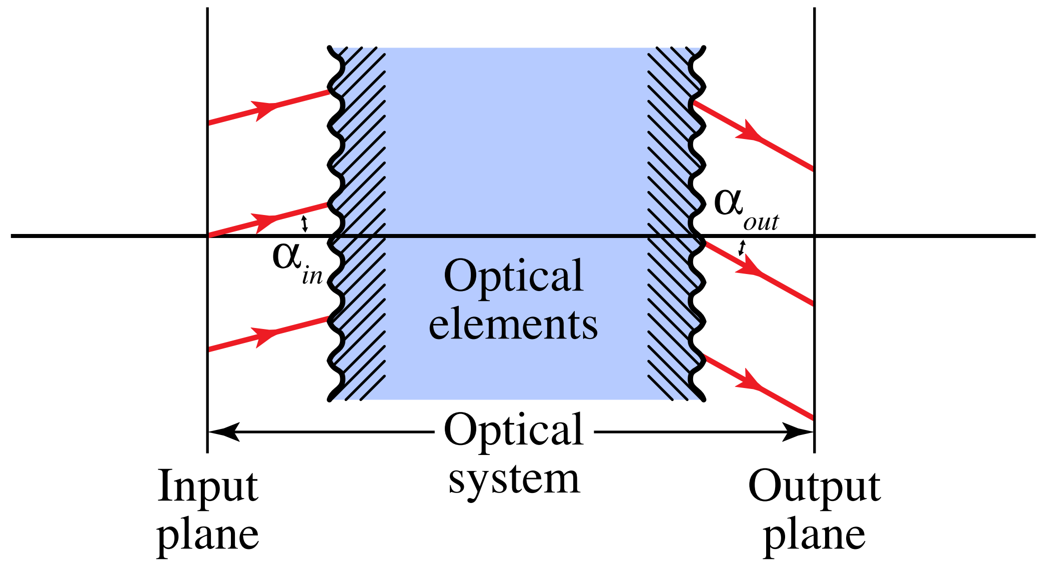
$$\alpha_{out} = D\alpha_{in}$$

System Ray-Transfer Matrix

Let's see what happens when $C = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in} + B\alpha_{in}$$
$$\alpha_{out} = D\alpha_{in}$$



When $C = 0$, collimated light at the input plane is collimated light at the output plane, but the angle with the optical axis is different. This is a telescopic arrangement, with an angular magnification of $D = \alpha_{out}/\alpha_{in}$.

System Ray-Transfer Matrix

Let's see what happens when $B = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in}$$

$$\alpha_{out} = Cy_{in} + D\alpha_{in}$$

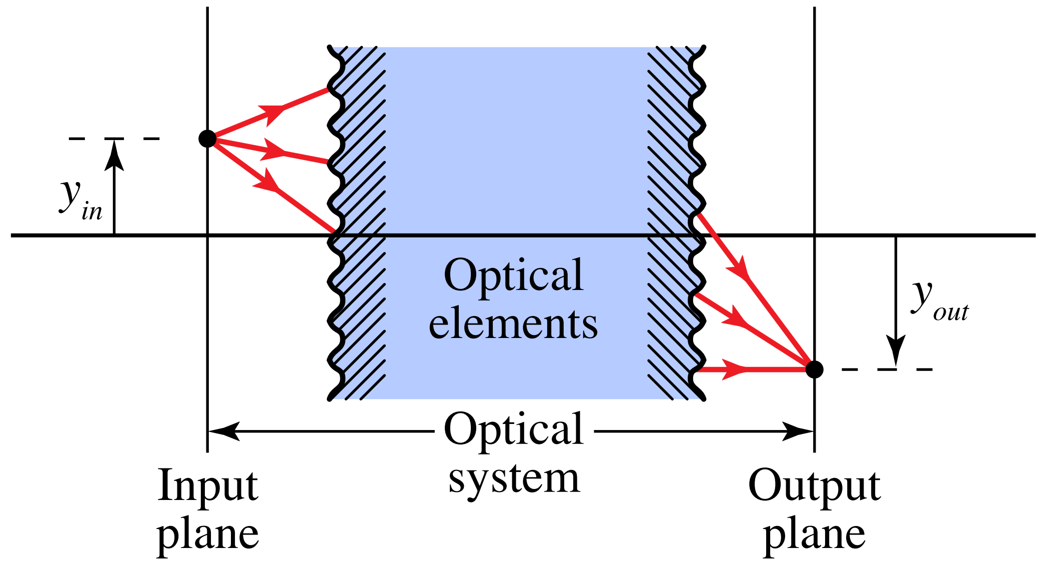
System Ray-Transfer Matrix

Let's see what happens when $B = 0$.

$$\begin{bmatrix} y_{out} \\ \alpha_{out} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} y_{in} \\ \alpha_{in} \end{bmatrix}$$

$$y_{out} = Ay_{in}$$

$$\alpha_{out} = Cy_{in} + D\alpha_{in}$$



When $B = 0$, the input and output planes are object and image planes, respectively, and the transverse magnification of the system $m = A$.

System Ray-Transfer Matrix – Summary

When $D = 0$, the input plane for the optical system is the input focal plane.

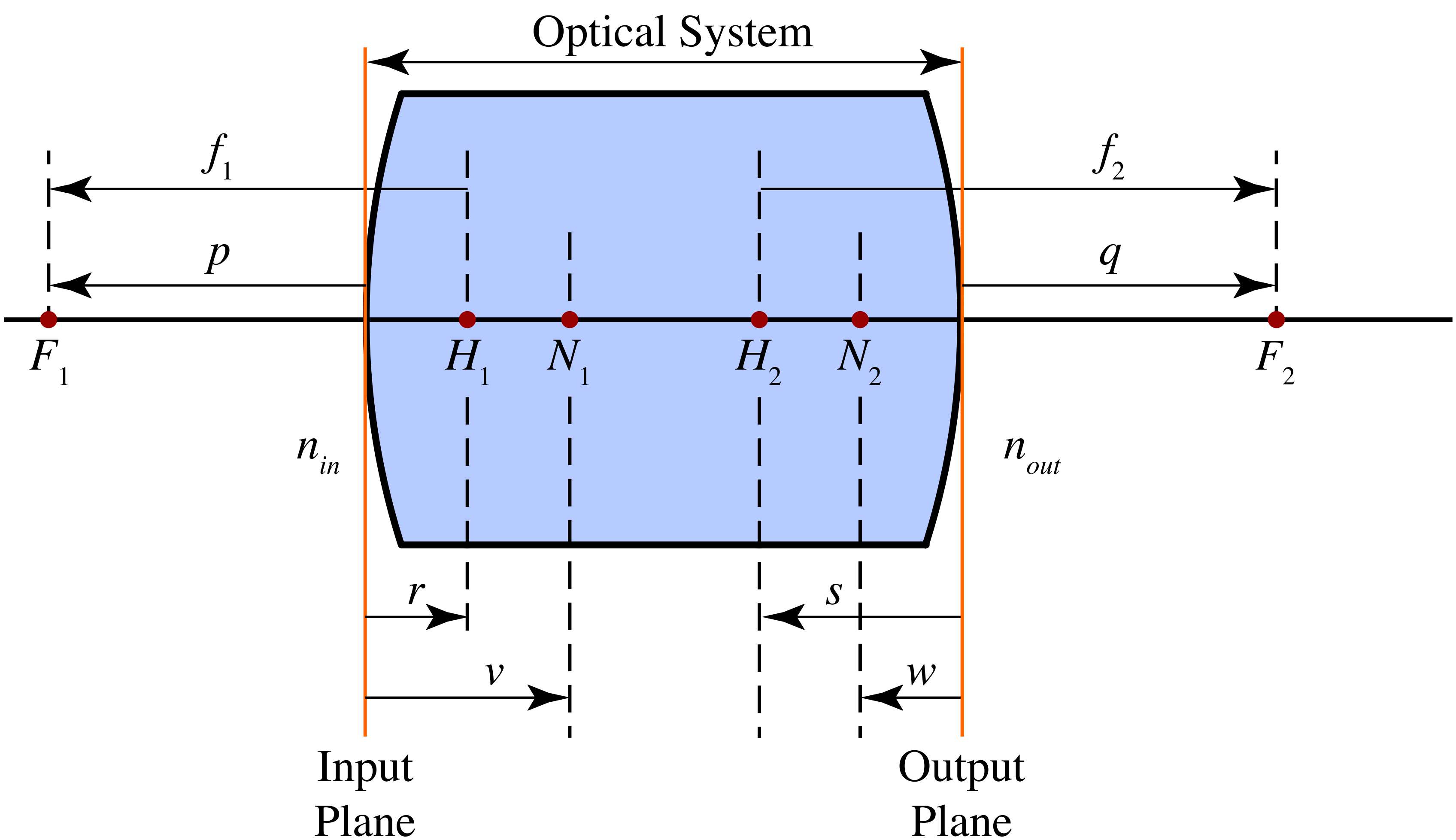
When $A = 0$, the output plane for the optical system is the output focal plane.

When $B = 0$, the input and output planes are object and image planes, respectively, and the transverse magnification of the system $m = A$.

When $C = 0$, collimated light at the input plane is collimated light at the output plane, but the angle with the optical axis is different. This is a telescopic arrangement, with an angular magnification of $D = \alpha_{out} / \alpha_{in}$.

System Ray-Transfer Matrix

The matrix elements of the system matrix can be analyzed to determine the cardinal points and planes of an optical system.



Cardinal Points

$$p = \frac{D}{C} \qquad F_1$$

$$q = -\frac{A}{C} \qquad F_2$$

$$r = \frac{D - n_{in}/n_{out}}{C} \qquad H_1$$

$$s = \frac{1 - A}{C} \qquad H_2$$

$$v = \frac{D - 1}{C} \qquad N_1$$

$$w = \frac{n_{in}/n_{out} - A}{C} \qquad N_2$$

Located relative to input (1) and output (2) reference planes.

+ values are to the right of the plane.
– values are to the left of the plane.

$$f_1 = p - r = \frac{n_{in}/n_{out}}{C} \qquad F_1$$

$$f_2 = q - s = -\frac{1}{C} \qquad F_2$$

Located relative to principal planes.