1–D Wave Equation

Let’s look at a wave in a stretched string. We can assume small angles for most situations.
Looking at the motion in the $x$-direction

$$\sum F_x = m a_x$$
Looking at the motion in the $x$-direction

$$\sum F_x = ma_x$$

$$ (+F_R \cos \theta_R ) + ( -F_L \cos \theta_L ) = m(0)$$
Looking at the motion in the $x$-direction

$$\sum F_x = ma_x$$

$$(+F_R \cos \theta_R) + (-F_L \cos \theta_L) = m(0)$$

$$F_R(1) - F_L(1) = 0$$
Looking at the motion in the $x$-direction

$$\sum F_x = m a_x$$

$$(+F_R \cos \theta_R) + (-F_L \cos \theta_L) = m(0)$$

$$F_R (1) - F_L (1) = 0$$

$$F_R = F_L = F_T$$
Looking at the motion in the $x$-direction

$$\sum F_x = m a_x$$

$$(+F_R \cos \theta_R) + (-F_L \cos \theta_L) = m(0)$$

$$F_R(1) - F_L(1) = 0$$

$$F_R = F_L = F_T$$

This is what we have seen before with strings; however, $F_R \neq F_L$ in general, e.g., when $a_x \neq 0$ and $m_{\text{string}} \neq 0$. 
Looking at the motion in the $y$-direction

$$\sum F_y = m a_y$$
Looking at the motion in the $y$-direction

$$\sum F_y = m a_y$$

$$\left( F_T \sin \theta_R \right) + \left( -F_T \sin \theta_L \right) = m \frac{\partial^2 y}{\partial t^2}$$
Looking at the motion in the $y$-direction where $\mu$ is the linear mass density.

\[
\sum F_y = ma_y
\]

\[
(F_T \sin \theta_R) + (-F_T \sin \theta_L) = m \frac{\partial^2 y}{\partial t^2}
\]

\[
F_T \left( \frac{\partial y(x + \Delta x)}{\partial x} - \frac{\partial y(x)}{\partial x} \right) = \mu \Delta x \frac{\partial^2 y}{\partial t^2}
\]

where $\mu$ is the linear mass density.
\[ \frac{\partial y(x + \Delta x)}{\partial x} - \frac{\partial y(x)}{\partial x} = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2} \]
\[
\left( \frac{\partial y(x + \Delta x)}{\partial x} - \frac{\partial y(x)}{\partial x} \right) = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2}
\]

In the limit as \(\Delta x \to 0\)

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F_T} \frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad v = \sqrt{\frac{F_T}{\mu}}
\]