I Transformations

1. Below are several examples of $\boldsymbol{A}\boldsymbol{r}_1 = \boldsymbol{r}_2$, where $\boldsymbol{A}$ is an active transformation matrix, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are 2-D vectors in the *x*-*y* plane. You are given $\boldsymbol{A}$ and $\boldsymbol{r}_1$ and you are to find $\boldsymbol{r}_2$. In each case, note whether $\boldsymbol{r}_2$ is parallel to $\boldsymbol{r}_1$ or not.

   (a) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$ parallel/not parallel

   (b) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$ parallel/not parallel

   (c) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$ parallel/not parallel

   (c) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \quad \\ \quad \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$ parallel/not parallel

   (In this case, pick your own vector $\boldsymbol{r}_1$.)

2. How can you tell if two vectors are parallel?

3. In general, does matrix multiplication change the magnitude of a vector? The direction? Explain your reasoning.

✓ **CHECK** your results with your instructor!

II Rotations and Reflections

1. In Section 7 in Chapter 3 of *Boas*, you read about rotation and reflection transformation matrices.

   (a) In general, does a rotation or reflection transformation matrix change the magnitude of a vector? The direction? Explain your reasoning.

   (b) Is the matrix from Section **I**, $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, a rotation or reflection transformation matrix?

   Support your answer by 1) using your results from Section **I**, and 2) assuming that you haven't done Section **I**.

2. Consider the following two matrices.

   (i) $\boldsymbol{A}_1 = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ (ii) $\boldsymbol{A}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

   (a) Are either of the matrices a rotation matrix? If so, find the rotation angle.

   (b) Are either of the matrices a reflection matrix? If so, find the line of reflection.

   (c) By examining matrices $\boldsymbol{A}, \boldsymbol{A}_1$, and $\boldsymbol{A}_2$ carefully, do you think it would be possible to multiply matrix $\boldsymbol{A}$ by a constant to make it a rotation or reflection matrix? If so, what would that constant be? If not, why not?

✓ **CHECK** your results with your instructor!

III Visualizing a Transformation

1. Consider the following active transformation matrix.

$$\boldsymbol{A}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   Pick 2 or 3 of your own vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$. How does matrix $\boldsymbol{A}_3$ transform your vectors? Does it change the magnitude of any of your vectors? Does it change the direction? Is $\boldsymbol{A}_3$ a rotation or reflection matrix?

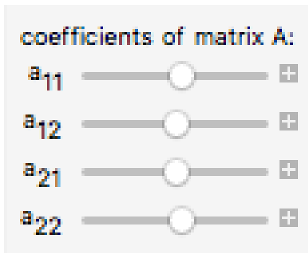
2. Download the Mathematica notebook "2-D Transformation Matrices" from the class resources page, and open it with Mathematica. Don't make any changes at this point; we'll get to that in just a minute.

   The notebook will take a vector $\boldsymbol{u}$, displayed in red on the graph of the x-y plane, and multiply it by an active transformation matrix $\boldsymbol{A}$. The resulting vector is displayed as blue on the graph. You can see the elements of the matrix $\boldsymbol{A}$ in the upper right corner of the graph, as well as the components of the two vectors.

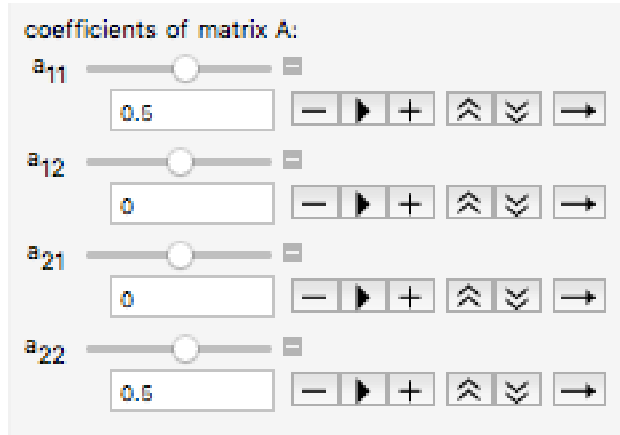
   (a) The default transformation matrix in the notebook is $\boldsymbol{A} = \boldsymbol{A}_3$, given in question 1 above. How does the matrix $\boldsymbol{A}$ transform the vector $\boldsymbol{u} = (1, 0)$? Does it change the magnitude of $\boldsymbol{u}$? The direction of $\boldsymbol{u}$? Is this consistent with your findings in question 1 above?

   (b) Click on the tip of the red $\boldsymbol{u}$ vector, and drag it to create a different $\boldsymbol{u}$ vector. How does the matrix $\boldsymbol{A}$ transform the new vector you created? Does it change the magnitude of $\boldsymbol{u}$? The direction of $\boldsymbol{u}$? Is this consistent with your findings in question 1 above?

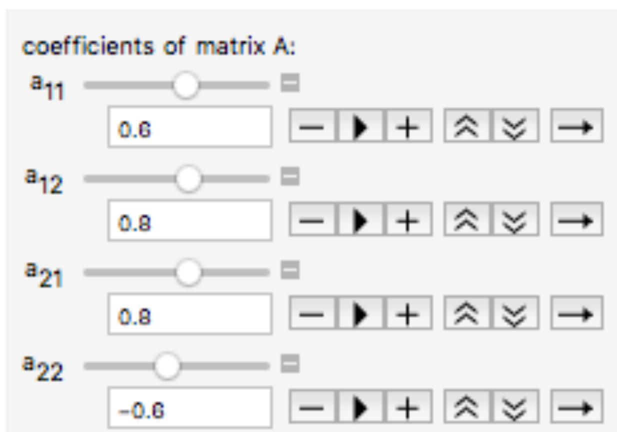
3. If you look in the upper left corner of the graph, you can see that you can change the individual elements of the ***A*** matrix. The best way to do so is to click on each of the + icons to the right of the sliders, and enter the numbers into the boxes below each slider.



$\Rightarrow$

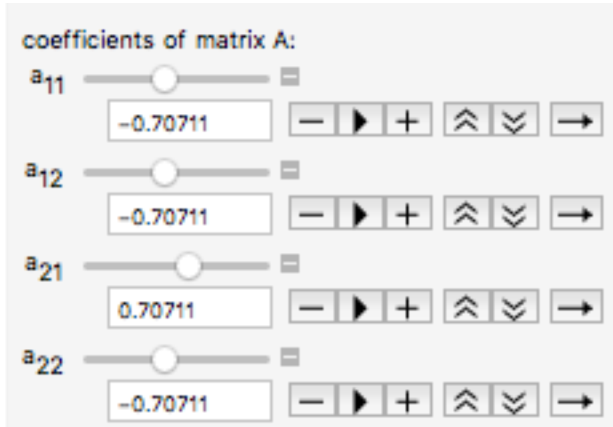


(a) Enter the values of matrix $\boldsymbol{A}_1$ from question 2 (Section **II**) in the boxes. It should look like the image shown below. **Hint and warning**: After typing in “0.6” for a_{11}, you can press “tab” to get to the box for a_{12}. However, don’t press “Return” or “Enter” after entering a number. Doing so will create a duplicate graph. After entering your value for a_{22}, just press “tab” one last time.



Drag the tip of the red ***u*** vector around to see how matrix $\boldsymbol{A}_1$ transforms different vectors. Does this match with what you found in question 2 (a) of Section **II**?

(b) Now enter the values of matrix $\boldsymbol{A}_2$ from question 2 (Section **II**) in the boxes. It should look like the image shown below.



Again, drag the tip of the red $\boldsymbol{u}$ vector around to see how matrix $\boldsymbol{A}_2$ transforms different vectors. Does this match with what you found in question 2 (b) of Section **II**?

IV Eigenvalues and Eigenvectors

1. Consider the following active transformation matrix.

$$\boldsymbol{A}_4 = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$$

(a) Following the method modeled at the beginning of Section 11 in Chapter 3 of *Boas*, find the eigenvalues of this transformation.

(b) Now find the eigenvectors for each of the eigenvalues you found in part (a).

(c) Enter the values of matrix $\boldsymbol{A}_4$ into the "2-D Transformation Matrices" Mathematica notebook. Move the tip of the red ***u*** vector to create one of your eigenvectors from part (b). How does the transformation matrix $\boldsymbol{A}_4$ transform the ***u*** vector? Does it change the direction of ***u***? The magnitude of ***u***? Is it consistent with the eigenvalue you found in part (a)?

(d) Move the tip of the red ***u*** vector to create a new vector that is *parallel* to the eigenvector you chose in part (c), but of a different length. How does the transformation matrix $\boldsymbol{A}_4$ transform this new ***u*** vector? Could this new vector also be considered an eigenvector?

(e) Repeat parts (c) and (d) with your other eigenvector. How do the directions of the two eigenvectors compare with each other?

(f) Move the tip of the red ***u*** vector to create vectors that aren't parallel to either of the eigenvectors. How does the transformation matrix $\boldsymbol{A}_4$ transform these vectors? Are the magnitudes and/or directions of the vectors changed?

2. Using only the “2-D Transformation Matrices” Mathematica notebook, find the eigenvectors and eigenvalues for each of the following active transformation matrices. How do the directions of the two eigenvectors compare with each other in each case?

   (g) $\boldsymbol{A}_5 = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$.

   (h) $\boldsymbol{A}_6 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$.

   (i) From Section **I**, $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.