I: Generalized coordinates, intuitively

Consider a simple projectile (no air resistance, constant mass, point particle) shot at a 45-degree angle with an initial speed of v_1. If you were to solve for the particle's maximum height or time of flight, you would break the motion into two coordinates, horizontal distance and vertical height, and solve each somewhat independently.

A. Consider a block sliding down straight down an inclined plane. As above, consider it to be a point particle.

1. Sketch the system, including the coordinate system you would use to think about the situation.

2. I'm guessing you chose a 2-dimensional coordinate system. (If you didn't, answer as if…)

   a. Why didn't you include a third dimension in your coordinate system?

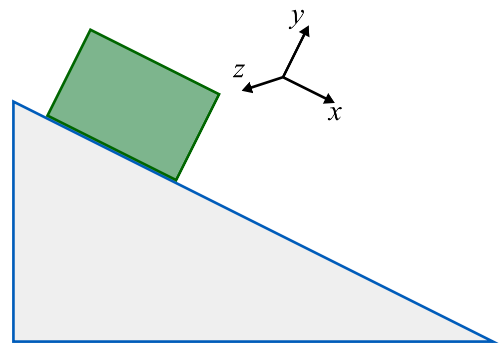
   b. How many coordinates are actually needed to describe the motion of (but not the forces on) the block? Think about the motion as simply as possible. What would you choose for this coordinate?

   c. Why **did** you include the *second* coordinate? Explain.

No, really! Read the following paragraph:

The coordinate you chose to represent the motion of the block is called a *generalized coordinate.* When you describe a system in terms of generalized coordinates, you pick the coordinates with the goal of completely describing the motion of the system in the fewest number of coordinates. In the Lagrangian formulation, the kinetic and potential energies are written in terms of the generalized coordinates of the system and the equation of motion is found from these. Describing the forces isn't necessary, so you can use fewer coordinates.

II: Generalized coordinates, formally



At right, I show the block from section I with a fixed, 3-D, Cartesian coordinate system, where z points out of the page.

In section I, you used your intuitions about the block-and-plane system to figure out that only one coordinate was needed to describe the motion of the system. The formal method for finding the number of generalized coordinates needed is to first find the total number of degrees of freedom of the system and then to subtract the number of constraints on the system.

A. In the block example, how many ways could the block potentially move in space? Ignore the incline for a moment.

B. Obviously, the incline can't be ignored and places a constraint on the motion of the block. Equations of constraint are derived from the physical situation and allow us to remove unnecessary coordinates. They can be due to physical limitations on the motion of objects in the system or can be chosen simply because it is convenient to do so.

   1: What constraint does the ramp impose on the block? Describe it in words and write an equation.

   2: Are there any other constraints, either physical or convenient? Write an equation for each additional constraint you find.

C. How many generalized coordinates are needed? Does this analysis agree with your informal reasoning of A.2.b? If not, reconcile your answers. By "reconcile," I mean "make consistent" and "come to an understanding about the source of the inconsistency."

D. Consider a new system. Could it have an equal number of degrees of freedom and equations of constraint? More equations of constraint than degrees of freedom? Explain the physical situation in each case.

E. Consider solving systems of linear equations. How are the relationships between the numbers of equations and the numbers of unknowns consistent (or different) from your answer to D? (discuss with your group).

III: A movable ramp

In the previous example, it was fairly clear how to choose the generalized coordinate – we just took away two coordinates from an already established coordinate system. However, the choice of coordinates is not always so straightforward for more complicated systems.

Think again about the block sliding down the ramp. Now, allow the ramp to move as well.

A. Sketch the system. Describe what will happen to the block and the ramp if the block is released from rest at the top of the ramp. (Describe the motion qualitatively and explain the causes.)

B. Decide how many generalized coordinates are now needed to describe the system. (Hint: Think about each object individually.) Use the intuitive method of Section I or the formal method of Section II. Use both, if possible.

C. It seems reasonable to keep the "down the ramp" coordinate as one of the generalized coordinates. What would you choose for the other coordinate(s)? (Which object have you not yet described?)

D. Generalized coordinates are different from fixed coordinates in several ways.

   1: Do your coordinates have the same origin? If not, describe the origin of each coordinate. Do you know of other coordinate systems with the same property?

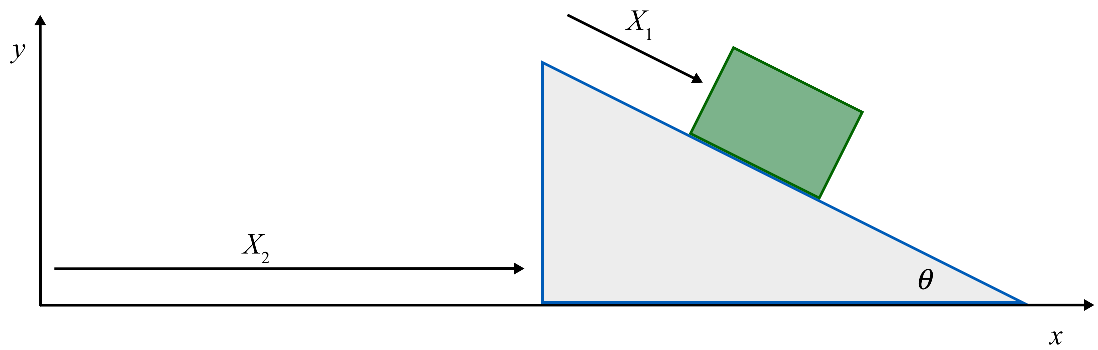
   2: Are your coordinates orthogonal? Do you know of other coordinate systems with the same property?

With your group, speculate on why the Newtonian (force) formulation of mechanics uses coordinates that have a common origin and are orthogonal, and the Lagrangian (energy) formulation need not. Discuss your thoughts with your instructor.

Please use a separate sheet of paper to show your work.

Usually, after generalized coordinates are chosen for a system, you write the kinetic and potential energies of the system in terms of the coordinates. Sometimes this is simple; other times it can be tricky. I have drawn the generalized coordinates traditionally chosen for this problem.

The variable X_2 points from an arbitrary point to the left edge of the incline, while X_1 points down the incline from the edge. The incline is at an angle θ above horizontal.



A. Consider the ramp. Using the generalized coordinates X_1 and X_2,

1. What is its kinetic energy? Explain your answer.

2. What is its potential energy? Explain your answer.

B. Consider the block on the ramp. Both generalized coordinates affect the motion of the block. Sometimes, it is possible to write the kinetic energy of the block "by inspection". For this problem, though, do the often simpler (though longer) following procedure:

1. First, express the x and y positions of the block in terms of the generalized coordinates.

2. Now, differentiate to find the x and y components of velocity in terms of the generalized coordinates.

3. Next, find the kinetic and potential energy of the block in terms of the external coordinate system. Recall that $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

4. Finally, using the equations you wrote in (1) and (2), rewrite the energies in terms of the generalized coordinates.

C. Consider the total system. Write expressions for the total kinetic and total potential energies of the system in generalized coordinates. Compare your answers to those given in the text in example 7.5.