In chapters 1, 2, 3, and 5, you used Newton's Laws to find the equations of motion for a system. The Lagrangian method is an alternative for finding the equations of motion. The Lagrangian is defined as:

$$L = T - U$$

where T is the kinetic energy of the system and U is the potential energy.

For conservative systems, the Lagrangian equation of motion is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

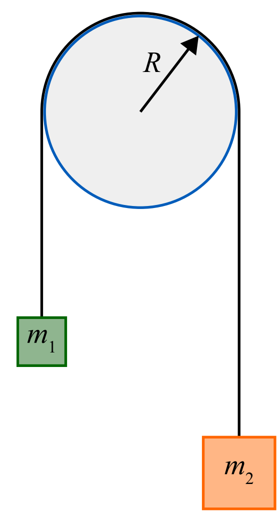
where q_i represents a particular generalized coordinate, and $\dot{q}_i$ is $\frac{dq_i}{dt}$. By using this equation separately for each generalized coordinate in a system, you can fully describe the motion of that system. For example, if the Lagrangian has three generalized coordinates, you will end up with three equations of motion.

I: Atwood's machine – Generalized coordinates

Consider the Atwood's machine – two blocks with unequal masses connected by a string of length L hung from a massless pulley of radius R. We want to find the equation of motion for the system.

A. Describing the system:

1. Describe in words what will happen to this system if $m_2 > m_1$.



2. Write an equation that relates the position of m_1 to the position of m_2. This is an *equation of constraint*, since it mathematically describes a constraint on the system. (Is this constraint physical or convenient?)

   a. Using your result above, write an equation that relates the velocity of m_1 to the velocity of m_2.

   b. Keep going and write an equation that relates the accelerations of the two masses.

B. Choosing the appropriate generalized coordinates for the Atwood's machine:

1. How many generalized coordinates do you need for this system?

2. I'm going to answer the previous question, with the assumption that you've thought about it yourself! (If you haven't, do so!) The Atwood's machine can be described by just one generalized coordinate. However, there are at least three reasonable choices for this coordinate. Find two. Be sure to discuss where each coordinate is zero, and how each coordinate increases or decreases. (Sketches might be useful here!)

3. As a group, decide which of your possible generalized coordinates to use. Will your choice of coordinate affect the speed of m_1?

II: Atwood's machine – Finding the Lagrangian equation of motion

A. First, write the energies in terms of the generalized coordinate:

1. What is the kinetic energy? (*Hint:* Find T for each block, then add.)

2. What is the potential energy? (Make sure that the potential energy for each block decreases as that mass moves down.)

B. Write the Lagrangian $L = T - U$.

C. The next step is to substitute the Lagrangian into the equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

1. Rewrite the Lagrangian in terms of the generalized coordinate you chose.

2. In your Lagrangian, what symbol did you use in place of q? What quantity does that symbol represent?

3. What symbol did you use in place of $\dot{q}$? What quantity does it represent?

D. Now, to the math! I have written each term with q's; you use whatever generalized coordinate you chose.

$$\frac{\partial L}{\partial q} =$$

$$\frac{\partial L}{\partial \dot{q}} =$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) =$$

Combine the terms to create the equation of motion:

E. When solving an equation, it's always critical to think of limiting cases that tell you about cases you can easily think about. This is one way of making physical sense of the mathematics.

1: What would you expect to happen if $m_1 = m_2$? What does your equation predict?

2: How about for $m_1 = 0$? $m_2 = 0$?

III: Thinking about Lagrangians

Discuss the following questions with your group. They are exploratory and perhaps don't have right and wrong answers.

A. You probably solved the Atwood's machine using Newton's Laws in your introductory mechanics class. Reach back to that memory and consider the many ways you have of solving the problem.

   1. Now that you have used two methods to solve Atwood's machine, which would you use if you saw it on an exam? Explain.

   2. What are the advantages of the Newtonian method?

   3. What are the advantages of the Lagrangian method?

B. If you prefer to use the Newtonian method for this problem, can you think of a problem that might be easier using the Lagrangian method? Describe it.

C. If you prefer to use the Lagrangian method for this problem, can you think of a problem that might be easier using the Newtonian method? Describe it.

Please use a separate sheet of paper to show your work.

The goal of this activity is to find the equation of motion for the simple pendulum using both the Newtonian and Lagrangian methods.

A. Starting from Newton's Second Law, use Newtonian methods to develop the equation of motion for the simple pendulum,

$$ml\ddot{\theta} = -mg\sin(\theta) .$$

B. Use the Lagrangian method developed in the tutorial to find the equation of motion for the simple pendulum:

1. How many generalized coordinates are needed? What is/are the most appropriate coordinate(s)?

2. Write the kinetic and potential energies in terms of the generalized coordinate. If you get stuck, try writing the energies in x and y coordinates, then convert to your generalized coordinate(s), as in the tutorial *Generalized Coordinates.*

3. Write the Lagrangian $L = T - U$.

4. Find the equation of motion using $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.