Math

Remember to *present* your solutions to the problem in words. Another student should be able to look at your homework page and be able to figure out what the question was asking without looking at this sheet. please show your work and explain your reasoning. I will grade for clarity of explanation as much as I do for mere "correctness of final answer"!

1) Vector decomposition (Conceptual)

Consider an arbitrary vector $\vec{A}$; and let the unit vector $\hat{e}$ point along some fixed coordinate direction. Using words and figures, explain the geometrical significance of the two terms in the following vector expansion: (You do not have to prove this relation, you just have to explain what it means!)

$$\vec{A} = \hat{e}\left(\vec{A}\cdot\hat{e}\right) + \hat{e}\times\left(\vec{A}\times\hat{e}\right) \tag{1}$$

Extra credit

Use your knowledge of the dot product to figure out the angle between the diagonal of one face of a cube and the "body diagonal" of the cube. *Hint: Let your cube have side length one, with one corner at the origin. Can you write down simple expressions for the vectors that represent a face diagonal, and the body diagonal*?

2) Mass density of a star

(a) A very simplistic model of a gaseous star might give you the mass density, ρ as a function of radius, r as

$$\rho(r) = \begin{cases} \rho_0 e^{-r/H} & r < R \\ 0 & r \geq R \end{cases} \tag{2}$$

Just looking at the expression, try to give some simple, physical interpretation of the three parameters ρ_0, H, and R. What are their units? What do they mean, or tell you, physically?

(b) Make a simple sketch of the mass density as a function of radius. (Your axes, where possible, should show how ρ_0, H, and R are involved)

(c) An even more simplistic model might instead give

$$\rho(r) = \begin{cases} c_0/r & r < R \\ 0 & r \geq R \end{cases} \tag{3}$$

In this case, can you compute the total mass of the star in terms of the two parameters c_0 and R? (If so, do it!) What are the units of c_0?

3) Integration review

A couple of math/calculus review questions

(a)

$$\int \frac{4x}{\left(x^4 + 2(bx)^2 + b^4\right)^{3/2}}\,dx \qquad (4)$$

(where b is a known constant. Note: this is an indefinite integral.)

(b)

$$\frac{d}{dy}\int_c^y f(x)\,dx \qquad (5)$$

(where $f(x)$ is some given, known (well behaved) function of x, and c is a constant.)

(c)

$$\frac{d}{dy}\int_0^2 (\log x + \log y)\,dx \qquad (6)$$

4) Practicing multi-dimensional integrals

Practice, practice: I have noticed some difficulties in evaluating multi-dimensional integrals. Since this is one of our class learning goals, I am giving you some basic problems to practice with.

(a) Consider a solid sphere, determined by the volume $x^2 + y^2 + z^2 \le R^2$, with a non-uniform density $\rho(x,y,z) = A|z|$. What are the units of A? Determine the total mass of the sphere in terms of A and R. (*Hints: Use spherical coordinates. Can you write z in terms of r, ϕ, and θ? Can you think of a clever use of symmetry and limits of integration to deal with that unpleasant absolute value*?).

(b) Consider a flat disk of radius R, with an areal mass density given in polar coordinates (it's flat, so we don't have any third "z" dimension) by $\sigma(r,\phi) = Ar^2 \sin^2(\phi)$, where A is a given constant. Describe (in words, and a simple sketch) this mass density, what does it look like physically? Now, determine the total mass of the thin disk in terms of A and R.

5) The joy of Taylor expansions

Taylor series are the single most important and common approximation technique used throughout physics. We need to keep practicing with them!

(a) Suppose $f(x) = 1/(1+x)$. Find an approximate expression for $f(x)$, going out to second order. Use it to estimate $f(0.1)$ and $f(5)$. In both cases, compare the exact result for $f(x)$ with your series approximation, going to first order (terms proportional to x) and also second order (x^2). Does your estimate improve at second order in both cases? Comment on the general criterion about x you would guess tells you when the series approach seems to be a fruitful one here.

(b) Now let $f(x)=(1-x)/(1+x)$. Taylor expand to approximate $f(x)$ out to second order. There are many ways to do this. First, use the usual formal formula for Taylor expansion. Another way is to take your series answer in part (a), and multiply it by $(1-x)$. Try it this way too, and use it to check yourself, both methods should agree to all orders! Which method do you prefer?

(c) Now let $f(x)=(a-x)/(a+x)$, where a is a constant with units, like $a = 5$ m . Taylor expand to second order. In this case, do NOT start "from scratch", blindly using the Taylor formula. Instead, factor out "a", so that it looks exactly like what you had in part (b), except the "thing you're expanding in" isn't x. What is it? For this part, what is now the general criterion on x that tells you when this series approach seems to be a fruitful one? Why? (And please do not say "x must be small" or "$x << 1$". Small compared to what? You cannot compare meters to numbers, or apples to oranges!)

(d) Taylor's equation 2.33 gives the velocity of a ball that has been dropped from rest with linear drag. Obtain an approximate expression for $v_y(t)$ in the case of small drag. Include terms to 2nd order (feel free to use the front flyleaf of your textbook to save some grief!) What is the physical meaning of the leading nonzero term in your approximate solution? What is your interpretation of the *sign* of the next term? As in the previous question, what exactly do we mean by "small drag" here – what exactly is small, compared to what, here?

6) The joy of complex numbers

Practice with complex numbers.

(a) If $z_1 = 1-\sqrt{3}i$, draw z_1 in the complex plane. Compute its real and imaginary parts and magnitude. Write z_1 as $Ae^{i\phi}$ and determine A and ϕ.

(b) If $z_2 = 1/(1-i)$, draw z_2 in the complex plane. Compute its real and imaginary parts and magnitude. Write z_2 as $Ae^{i\phi}$ and determine A and ϕ.

(c) If $z_3 = 0.5e^{-i\pi/6}$, what are the real and imaginary parts of $1/z_3^2$? (*Note that this is just a simple square, not an absolute value squared, so the answer does not need to come out purely real*!)

(d) Compute $Z = z_1^* z_3$ with z_1 and z_3 given in parts (a) and (c). Draw Z in the complex plane. Compute its real and imaginary parts and magnitude. Write Z as $Ae^{i\phi}$ and determine A and ϕ.

(e) Pick either one of the following two items to prove - whichever you prefer!
– Euler's theorem says that $e^{i\phi} = \cos\phi + i\sin\phi$. Call $z = e^{i\phi}$ and evaluate z^2 two ways to show the (very handy!) trig identities $\cos 2\phi = \cos^2\phi - \sin^2\phi$ and $\sin 2\phi = 2\sin\phi\cos\phi$.
– Use Euler's theorem to rewrite $\sin x$ in terms of complex exponentials, and then use that to show that $\int_0^{2\pi}\sin(n\phi)\sin(m\phi)\,d\phi$ vanishes if n and m are unequal integers. (*We'll make good use of this*!)

7) The joy of Delta functions

Evaluate the following integrals:

(a) $\int_0^\pi \sin(x)\delta(x-\pi/2)\,dx$

(b) $\int_0^3 (5t-2)\delta(2-t)\,dt$

(c) $\int_0^5 (t^2+1)\delta(t+3)\,dt$

(d) $\int_{-\infty}^{\infty} e^x\delta(3x)\,dx$

(e) If a particle experiences a force $F(t)$ of the form $F(t) = A\delta(t)$, with t the time, what are the units of A? Give a brief physical interpretation to this formula – what sort of physical force are we trying to represent here?