Remember to *present* your solutions to the problem in words. Another student should be able to look at your homework page and be able to figure out what the question was asking without looking at this sheet. please show your work and explain your reasoning. I will grade for clarity of explanation as much as I do for mere "correctness of final answer"!

Choose 5 of the following 11 problems to complete.

1) Computing line integrals of the magnetic field

The magnetic field inside a long current carrying wire is $\vec{B} = \left(\frac{B_0}{R}\right) r\hat{\phi}$, where R is the radius of the wire, and r is the distance from the center of the wire. Ampere's law relates the integrated magnetic field around any closed loop to the total current passing through the loop, $\oint \vec{B} \bullet d\vec{r} = \mu_0 I_{\text{through}}$. If we want to determine the current passing through the loops shown in Figure 1, we need to evaluate the line integral of $\vec{B} \bullet d\vec{r}$.

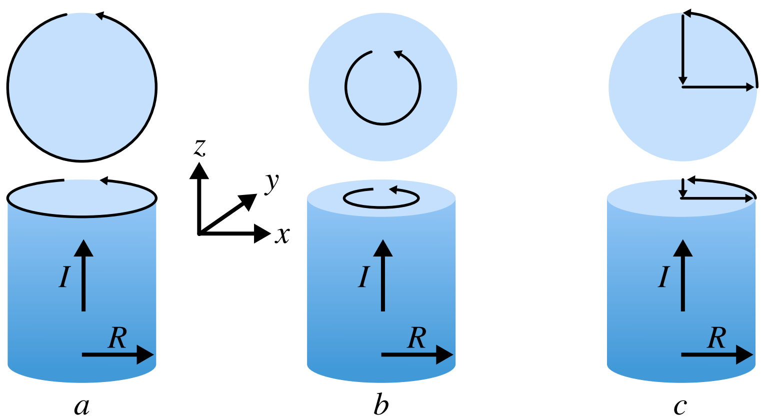


Figure 1

(a) Explicitly compute $\oint \vec{B} \bullet d\vec{r}$ along the full circle path of radius R, shown in Fig.1a. Use this to find I_{through}. Briefly, discuss the physical meaning of the sign of your answer. If I had asked you to integrate the other way around the circle, what sign(s) in your solution would have changed? Would your final conclusion regarding the direction of I have changed? Be clear about this – what does the sign of your integral tell you?

(b) Compute $\oint \vec{B} \bullet d\vec{r}$ along the path of radius r_0 (less than R) in Fig 1b. How does I_{through} compare with part (a)?

(c) Compute $\oint \vec{B} \bullet d\vec{r}$ along the quarter circle path in Fig 1c. Compare your answer to parts (a) and (b), and discuss. (What do you conclude about how the current is distributed through the wire?)

(d) Sketch, by hand, a vector plot of $\vec{B} = \left(\frac{B_0}{R}\right) r\hat{\phi}$. Rewrite $\vec{B}$ entirely in Cartesian coordinates, and then use the VectorPlot command in Mathematica to generate a plot of $\vec{B}$ to check your hand-drawn sketch.

2) Shrinking an orbit

A puck (mass m) on a frictionless air hockey table is attached to a cord passing through a hole in the surface as shown in the figure below. The puck is moving in a circle of radius r_1 with angular speed ω_1. The cord is then slowly pulled from below, shortening the radius to r_2.

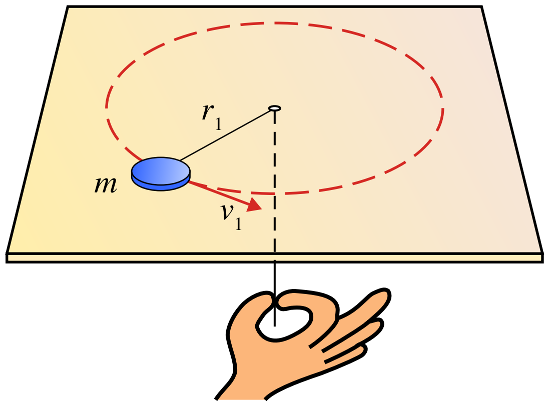


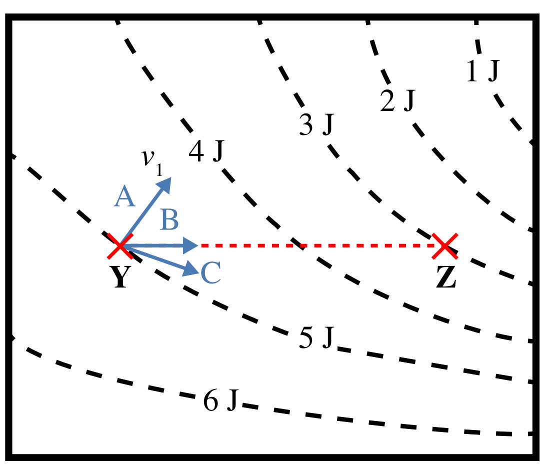
Figure 2

(a) What is angular velocity of the puck when the radius is r_2?

(b) Assuming that the string is pulled so slowly that we can approximate the puck's path by a circle of slowly shrinking radius, calculate the work done in moving the puck from r_1 to r_2. (Look back at Taylor Eq 1.48. "Slowly" means that $\dot{r}$ is tiny, as is the angular/rotational acceleration, so only the inward component of the tension force by the string will be important). Compare your answer with the puck's gain in kinetic energy, and comment briefly.

3) Choose an initial velocity

The diagram below shows a region of space. The dashed curves indicate positions of equal potential energy and are labeled with the value of the potential energy at that curve. Three vectors originating from point **Y** are also shown in blue. The vector **B** points directly from point **Y** to **Z.** In order for a particle to be launched from **Y** and reach **Z**, which vector represents best the initial velocity? Explain your reasoning. Then, make a (very crude, no calculations required!) sketch/guess of the *path* you expect the particle to follow if launched from point **Y** with initial velocity $\vec{v}$ given by vector **A**. (Explain your reasoning, briefly so I know what you were thinking about.)

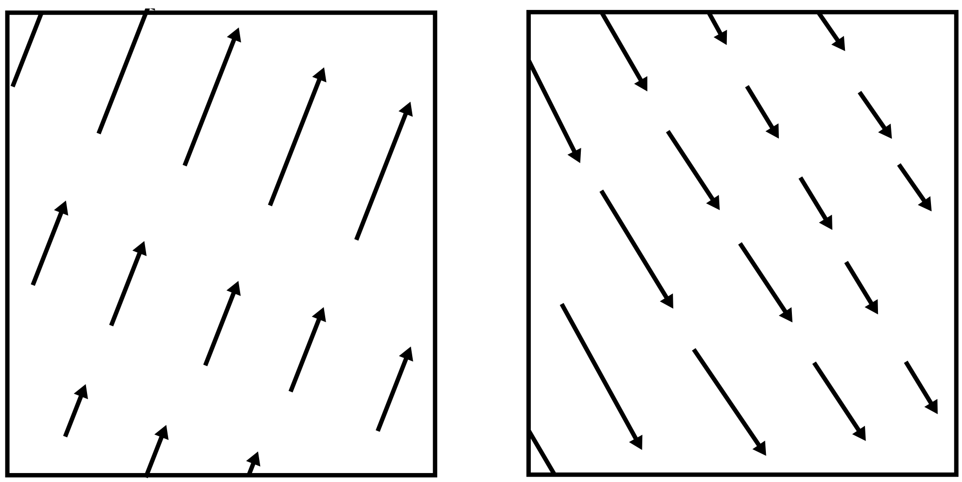


4) You're a collision investigator!

A driver traveling *downhill* slams on the brakes and skids 40 m on before hitting a parked car. You have been hired as a physics expert to help the insurance investigators decide if the driver had been traveling faster than the 25 MPH speed limit at the start of this event. The slope of the hill is 5 degrees. Assuming braking friction has the usual form μF^{norm}, what is the "critical value" of μ for which you would conclude the driver was speeding? Can you convince the investigators this driver was speeding, or do you need more information? (Do you need to know the mass of either vehicle? Road conditions?) While there are multiple ways to solve this problem, *please solve it using work and energy*.

5) Properties of conservative fields

Each diagram in the figure below depicts a force field in a region of space.



Case 1 Case 2

(a) For which force field can you identify a closed path over which $\oint \vec{F} \bullet d\vec{l} = 0$? For each such case, clearly indicate an appropriate path on the diagram.

(b) For each case indicate if $\vec{\nabla} \times \vec{F} = 0$ everywhere in the box? Also, for each case, could the force depicted in the diagram be conservative? Briefly, explain.

(c) For each case, is it possible to draw a self-consistent set of equipotential contours for that situation? If so: Draw them! (Each drawing should clearly show the correct shape of the contour lines, the correct relative spacing of the contours, and labels showing the regions that correspond to highest and lowest potential energy.) If not: Explain why drawing such contours is impossible.

6) Fictitious potential energy of a satellite

Consider the following expression for the potential energy of a satellite in a far away solar system.

$$U(x,y,z)=\left(2\tfrac{\mathrm{J}}{\mathrm{m}^2}\right)x^2-\left(1\tfrac{\mathrm{J}}{\mathrm{m}^2}\right)y^2-\left(1\tfrac{\mathrm{J}}{\mathrm{m}^2}\right)xy+\left(3\tfrac{\mathrm{J}}{\mathrm{m}}\right)z$$

(a) Compare the magnitude of the force on the satellite at point 1 (0,0,1) and point 2 (1,2,0).

(b) Is this a conservative force? Clearly justify your answer.

(c) For an arbitrary scalar function $h(x,y,z)$, evaluate the components of $\vec{\nabla}\times\vec{\nabla}h=0$ in Cartesian coordinates and show that the result is 0. Does this formal result relate to part (b)? (If not, why not? If so, briefly comment)

(d) Explicitly compute the work done by this force as you follow a path from the origin (0,0,0) to point 2, along a straight line. Then, check your answer very simply using the potential function given. Explain the idea of your check.

7) Time derivative of the total energy of a pendulum

A simple pendulum consists of a point object of mass m fixed to the end of a massless rod (of length l), whose other end is pivoted from the ceiling to let it swing freely in the vertical plane. The pendulum's position can be identified simply by its angle ϕ from the equilibrium position.

(a) Write the equation of motion for ϕ using Newton's second law. Assuming that the angle ϕ remains small throughout the motion, solve for $\phi(t)$ and show that the motion is periodic. What is the period of oscillation?

(b) Show that the pendulum's potential energy (measured from the equilibrium position) is $U(\phi)=A(1-\cos\phi)$. Find A in terms of m, g and l. Then, write down a formula for the *total energy* as a function of ϕ and $\dot{\phi}$.

(c) Show that by differentiating the energy with respect to t you can recover the equation of motion you found in (a). (What basic principle of physics are you using, here?)

(d) Picking simple values for parameters (*e.g.*, $m = l = 1$; $g = 10$), use your favorite computational environment (*e.g.*, NDSolve in Mathematica) to solve for $\phi(t)$ given a fairly small starting angle ϕ_1. Provide output that clearly shows that the period of oscillation is very close to the theoretical prediction from part (a). Repeat with $\phi_1 = 2\text{ rad}$ (which is quite large), and show that the period now deviates from ideal. (Does the period get *larger* or *smaller* as ϕ_1 increases?)

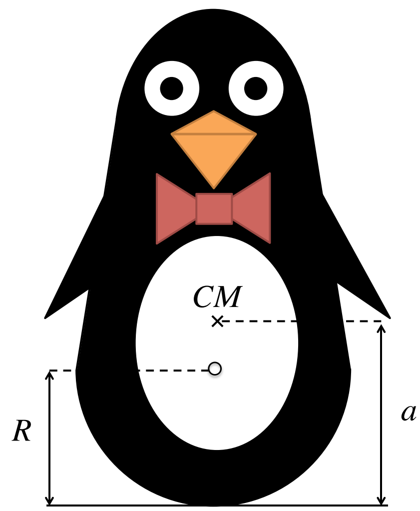
(e) Using your numerical solution to part (d) and your formulas for energy from part (b), generate a single plot which graphs KE, U, and total E as a function of time, for one period, in the case where $\phi_1 = 2\text{ rad}$ is NOT small. Briefly, comment! (Do we still conserve energy when we cannot use the small angle approximation anymore?)

8) Stability of the weeble

Look at the figure below, which shows a possible design for a child's toy. The designer (you!) wants to build a "weeble" (which wobbles but doesn't fall down). It is a single, solid object, the base shape is a perfect hemisphere (radius R) on the bottom, connected to the rest of the body (here, a penguin). The CM is shown – it is "a" above the base. You don't need to try to compute "a"; assume it is a given quantity! (As the designer, how might you control/change the position of "a"?)

(a) Write down the gravitational potential energy when the weeble is tipped an angle θ from the vertical, as a function of given quantities (m, a, R, g, and of course θ).

(b) Assuming the toy is released from rest, determine the condition(s) for any equilibrium point(s). Then consider stability: what values of (or relations between) "a" or "R" ensure that the weeble doesn't fall down? Explain. Can this toy work as desired?



9) A simple 1D nonlinear force

A particle is under the influence of a force $\vec{F} = k^2\left(-x + x^3/\alpha^2\right)\hat{x}$, where k and α are constants.

(a) What are the units of k and α ? Find $U(x)$ (assuming $U(0) = 0$) and sketch it. (Sketch here means hand-drawn, not plotted with Mathematica. Include all features you consider interesting in your sketch, including *e.g.* zero crossings, behavior at large x, "scales" of your axes, etc.)

(b) Find all equilibrium points and determine if they are stable or unstable.

(c) Qualitatively explain the motion of an object in this force field released from rest at $x = \alpha/2$. Then, qualitatively explain the motion of an object in this force field released from rest at $x = 3\alpha/2$

This potential, or small variants of it, occurs in various physics situations. This is a more accurate form of the true force on a pendulum than our usual approximation, can you see why? Another (famous!) case, albeit with the signs flipped, gives the "Higgs potential" in particle physics.

10) A Gaussian well

A particle (in 2D) has a potential energy function given by $U(\vec{r}) = -U_0 e^{-k(x^2+2y^2)} = -e^{-(x^2+2y^2)}$, (assume $U_0 = 1$ and $k = 1$ and have implicit correct SI metric units so that x and y are in meters, and U in Joules.)

(a) Make an equipotential plot of U ("ContourPlot" in Mathematica), and a 3D plot ("Plot3D" in Mathematica). Include a copy of both plots with your homework. Now stare at these images. Describe in words how you would interpret this potential physically. (For instance, is this attractive, repulsive, both, neither? Can you invent some physical system for which this might be a crude model? Would a particle be "bound" here?) Which is more informative for you – the contour plot or the 3D plot? Why? If you put a particle at, say, $\vec{r} = (1,1)$, use your intuition and the plots to describe very clearly in words (without calculation) the approximate direction of the force there.

(b) Analytically compute the force on the particle. Then, plot the force field ("VectorPlot" or "StreamPlot" in Mathematica. If you name your plots "p1 = ContourPlot", "p2=VectorPlot" and make sure the x and y range is the same, then you can plot TWO different plots on top of each other using "Show[p1,p2]".) Produce a single plot that shows the contours and the force field together. Now, what is the magnitude and direction of the force at $\vec{r} = (1,1)$? Discuss whether your plots agree with your expectation in part (a) (and resolve any discrepancies).

(c) What changes could you make to $U(\vec{r})$ to make the potential well *i*) centered at the point (1,1) instead of (0,0), *ii*) stronger?, *iii*) repulsive? (You might want to use Mathematica to visualize your claims here.)

11) Electric force and potential energy

Electrical force is given by $\vec{F} = q\vec{E}$, where $\vec{E}$ is the electric field. The electric force is conservative in "electrostatic" situations (but is *not* conservative in all situations!)

(a) Consider a laboratory setup in which an object with charge q sits in a field $\vec{E} = A(z^2 + a)\hat{z} = A(z^2 + 1)\hat{z}$, (assume $a = 1$ and has implicit correct SI metric units). Prove that the resulting force is conservative, and deduce a formula for the potential energy as a function of position of this charge.

(b) You release the charged object (it has mass m) at the origin, starting from rest. Describe its motion qualitatively. How fast is it going when it reaches a distance h from the starting point? If the object was in the shape of a bead (*same* mass m and charge q) threaded on a curved, frictionless, rigid wire, which started at the origin (0, 0, 0) and ended at some point (x_0, y_0, h) would it also have that same speed you just calculated, or not? (Why?)