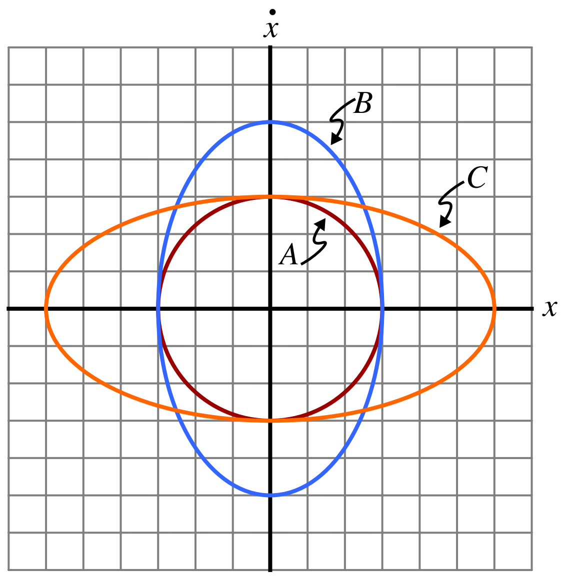
Remember to *present* your solutions to the problem in words. Another student should be able to look at your homework page and be able to figure out what the question was asking without looking at this sheet. Please show your work and explain your reasoning. I will grade for clarity of explanation as much as I do for mere "correctness of final answer"!

Choose 3 problems from problems 1–5 to complete.
Choose 1 problem from problems 6–7 to complete.
Choose 1 problem from problems 8–9 to complete.

1) Properties of harmonic oscillator phase space plots

Consider the phase space plots (A, B, and C) shown below



(a) Could all three plots correspond to the same simple harmonic oscillator (*i.e.,* same mass and same spring constant)? Explain why or why not.

(b) Which pair of plots could be used to show the effect of keeping the *total energy constant* but *increasing the spring constant* (while keeping the mass fixed)? Clearly indicate which plot would correspond to the smaller spring constant. Explain *without* performing any calculations.

(c) Suppose that plots A and C correspond to systems with springs with the same spring constant, but different masses. Do these two systems have the same total energy? If not, which one has more total energy? Explain how you can tell.

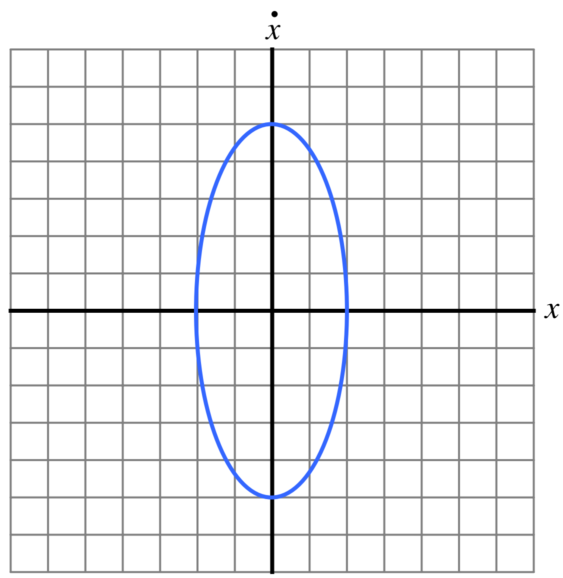
(d) Suppose that in the diagram each division along the position axis corresponds to 0.10 m; along the velocity axis, 0.10 m/s. What is the period of the oscillator shown by path B?

2) Sketching phase space diagrams

(a) A 4.0 kg object is connected to a spring with spring constant k = 1.0 N/m. At t = 0 the object is set into simple harmonic motion (no damping) by releasing it from rest at a point x = –1.0 m (*i.e.*, a meter to the left of the origin). Sketch an accurate phase space plot for the oscillator. Explain your reasoning and show your work. If the direction the particle follows around the plot is ambiguous, say so, otherwise draw an arrow to show how it moves around the phase space diagram.

(b) On the phase space trajectory you have drawn, label the point Q that represents the position and velocity of the oscillator one-quarter period after t = 0. Explain your reasoning. Also, on the phase space trajectory you drew, add a second trajectory (make it "dashed" so I can tell them apart) that shows the phase space plot of a system with the same total energy but smaller mass. Explain your reasoning briefly.

(c) Now consider an elliptical phase space trajectory for a different system, as shown in the figure below. Assuming each unit on the horizontal axis is 10 cm, and each unit along the vertical axis is 10 cm/s, and assuming a mass m = 1.0 kg, determine numerical values for the following quantities: i) angular frequency, ii) period, iii) total energy, and iv) spring constant.

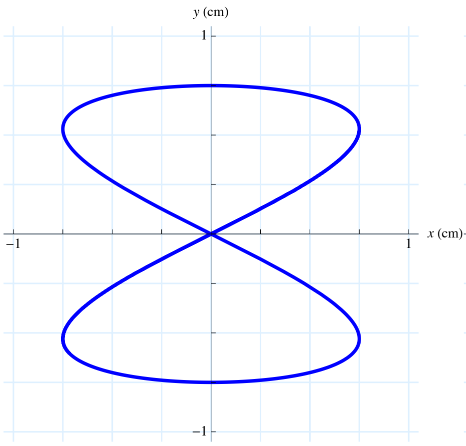


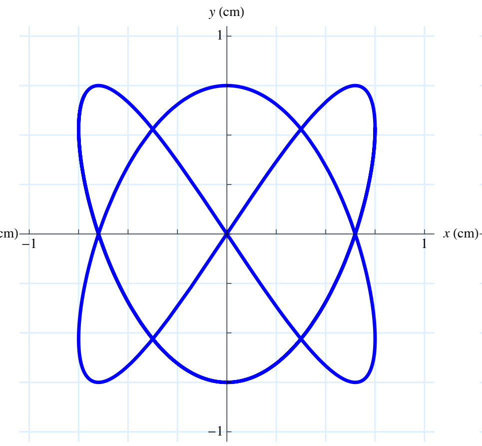
3) 2D non-isotropic oscillator with commensurate natural frequencies

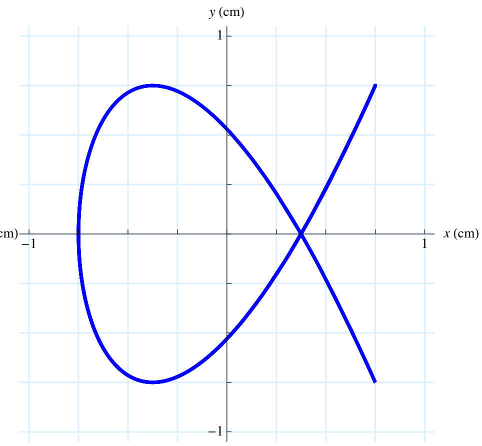
Consider the motion of a two-dimensional non-isotropic oscillator, in which the spring constant in the x–direction (k_x) is not equal to the spring constant in the y–direction (k_y). Each trajectory below depicts the possible motion of a unique oscillator. All three oscillators, however, share the property that the angular frequencies ω_x and ω_y for the motions along the x– and y–axes are *commensurate, i.e.,* that the angular frequencies satisfy the following relationship:

$$\frac{\omega_x}{n_x} = \frac{\omega_y}{n_y}, \text{ where } n_x \text{ and } n_y \text{ are integers.}$$

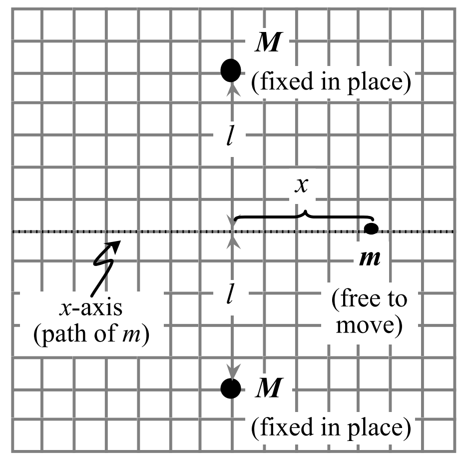
For each case shown on the next page, (i) determine whether ω_x is *greater than, less than,* or *equal to* ω_y, and (ii) determine the values n_x and n_y that satisfy the condition shown above. Explain.







4) **A gravitational oscillator?**



According to Newton's law of gravitation, the magnitude of the gravitational force exerted on one object with mass m_A by a second object with mass m_B is given by:

$$F^{grav} = G\frac{m_A m_B}{r^2}$$

where r is the distance between m_A and m_B, and G is the universal gravitational constant.

Two large solid spheres, each with mass M, are fixed in place a distance $2l$ apart, as shown. A small ball of mass m moves along the x-axis, shown as a dashed line. (Let $x = 0$ represent the point on the axis directly between the spheres, with $+x$ to the right.)

(a) Using Newton's law of gravitation and vector superposition, express the quantities listed below in terms of the given parameters (*i.e.,* in terms of *x, l, m, M, and G*). Show all work.

(i) the magnitude of the force on the ball (mass m) by one of the large spheres (mass M).

(ii) the magnitude of the net force on the ball by both large spheres.

(b) Use Newton's Second Law, $F_x^{net} = m\ddot{x}$, to write down a differential equation that governs the motion of the small ball.

(c) Is the net force exerted on the small ball a *restoring* force? Explain how you can tell from your differential equation above.

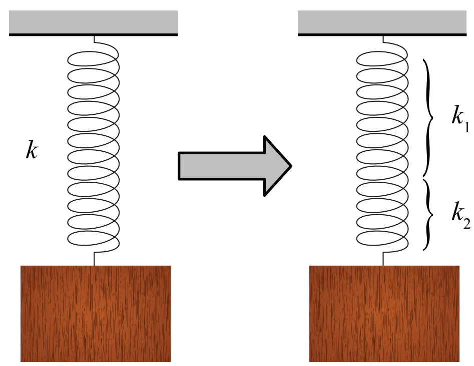
(d) Explain why it is *incorrect* to say that the small ball of mass m necessarily moves in *simple harmonic motion.*

(e) Under what limiting conditions for x is it possible to say that the motion of the mass m can be *approximated* as simple harmonic motion? (For large *x?* For small *x?* Large or small compared to *what?*) In this limiting case, determine an expression for the period of motion in terms of the given parameters. Explain your reasoning.

5) Parallel and series springs

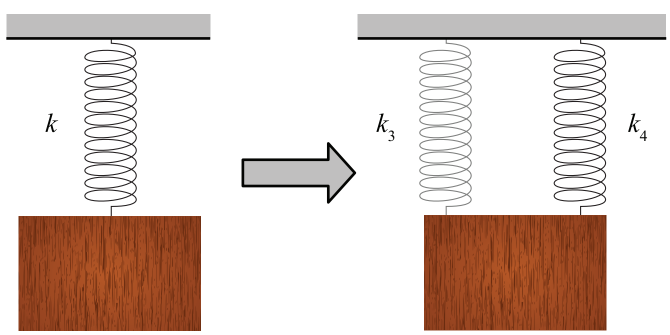
An ideal (massless) spring with force constant k is used to hang a crate from the ceiling, as shown. Suppose that the spring were treated as <u>two separate springs</u> (1 and 2) connected end-to-end. Treat each spring as having its own spring constant (k_1 or k_2) given by Hooke's law, $|F_i| = k_i x_i$, where $|F_i|$ is the magnitude of the force exerted on (or by) spring i when the length of that spring changes by an amount x_i (from its unstretched length).

(a) Relate the magnitudes of the forces F_1 and F_2 exerted by each spring individually, and the magnitude of the force F exerted on the crate. Explain. (Hint: Sketch separate free body diagrams for the springs and the crate.) Then, find the relationship between the stretches x_1 and x_2 of the individual springs and the stretch x of the original (single) spring. Explain your reasoning. Based on these results, determine an expression for the spring constant of the original spring (k) in terms of the spring constants (k_1 and k_2) of the two individual "spring pieces". Based on this problem, if you chop an ideal spring in half, what happens to its spring constant? Show all work.



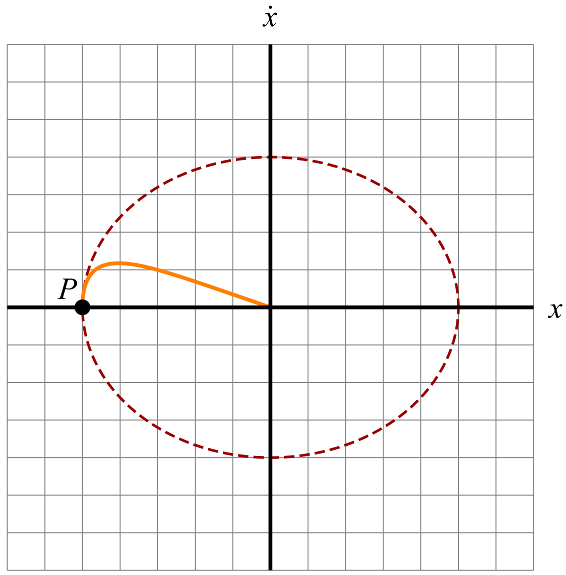
Crate (at rest)

(b) Now consider the case in which you replace the original spring with two new springs (3 and 4) that connect directly to the crate <u>and</u> to the ceiling. (See diagram.) Assume that each of the new springs has the same natural length as the original spring. Following the same logic as you did in part a, determine an expression for the spring constant k of the original spring in terms of the spring constants k_3 and k_4 of the new springs. Explain your reasoning.



6) Phase space of a damped oscillator

Shown below are the phase space plots for (i) a simple harmonic oscillator (dashed red) and (ii) the same oscillator with a retarding force applied (solid orange). Point *P* represents the initial conditions of the oscillator in both instances.



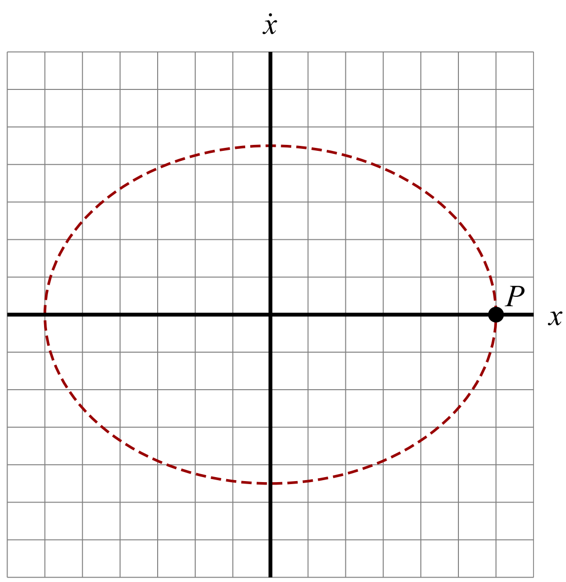
(a) Explain how you can tell that the damped oscillator is *not* underdamped.

(b) Is the damped oscillator *critically damped* or *overdamped?* Explain how you can tell.

(c) If you said in part (b) that the oscillator is {critically damped/overdamped}, then draw how the phase space plot would be different if the oscillator (starting at point *P*) were instead {overdamped/critically damped}. Explain your reasoning.

7) Adding linear damping to an undamped oscillator

The phase space trajectory of an undamped oscillator is shown below. In the diagram, each division along the position axis corresponds to 0.1 m; along the velocity axis, 0.10 m/s.



(a) What is the angular frequency ω_0 of the undamped oscillator? Explain how you can tell.

(b) A retarding force is now applied to the oscillator for which the damping constant is equal to $\beta = 0.0644\ \omega_0$ (using Taylor's notation, introduced in Eq. 5.28). By what factor does the amplitude change after a single oscillation? Show all work.

(c) On the basis of your results above, carefully sketch the phase space plot for the first cycle of the motion of the damped oscillator, starting at point *P*.

(d) If this system corresponded to a real pendulum, which effect of damping would be more noticeable – the change of the period or the decrease of the amplitude? Justify your opinion.

8) Picking out the transients

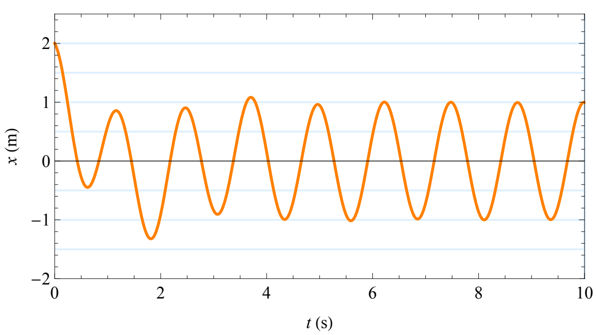
A harmonic oscillator with a restoring force $25m\alpha^2 x$ is subject to a damping force $4m\alpha\dot{x}$ and a sinusoidal driving force $F_0\cos(2\alpha t)$.

(a) Write down the differential equation that governs the motion of this oscillator.

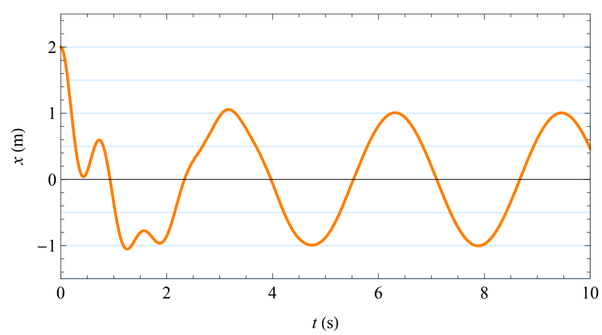
(b) Show that if the driving force were removed the oscillator would become *underdamped,* and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.

(c) For any damped oscillator that is driven by a sinusoidal external force, we know that the eventual (steady-state) motion is sinusoidal in nature. However, before the oscillator reaches steady state, its motion can be thought of as the algebraic sum of the steady-state motion plus a *transient* oscillatory motion whose amplitude dies exponentially with time. (In fact, the transient component of the motion (considered by itself) could accurately describe the motion of the same oscillator with the driving force turned off (but with the damping still present).)

(i) Each *x vs. t* graph below illustrates the actual motion (transient plus steady-state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion *greater than, less than,* or *equal to* that of the transient motion? Explain.



Graph 1



Graph 2

(ii) Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a–c of this problem. Explain your reasoning.

9) Leading and lagging in electrical oscillations

A series LRC circuit is connected across the terminals of an AC power supply that produces a voltage $V(t) = V_0\, e^{i\omega t}$. The "equation of motion" for the charge $q(t)$ across the capacitor is as follows:

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V_0\, e^{i\omega t}$$

The above differential equation will have a steady-state solution of the form:

$$q(t) = q_0\, e^{i(\omega t + \delta)}$$

[*Note:* The parameters q_0 and δ are actually functions of ω, the frequency of the AC power supply. However, in this problem you will not have to write out these functions in full.]

(a) In terms of q_0, ω, δ, and the relevant coefficients from the differential equation, write down the following functions:

(i) the potential difference $\Delta V_C(t)$ across the capacitor.

(ii) the potential difference $\Delta V_R(t)$ across the resistor.

(iii) the potential difference $\Delta V_L(t)$ across the inductor.

(b) Determine the smallest values of α, β, and γ (in radians) that satisfy the following Euler relations:

- $e^{i\alpha} = i$
- $e^{i\beta} = -1$
- $e^{i\gamma} = -i$

(c) Using your results from part b, rewrite the functions in part a so that each function can be written as a *positive real* number times a complex exponential. Use your rewritten functions to answer the following questions.

(i) What is the phase difference between $\Delta V_C(t)$ and $\Delta V_R(t)$*?* Do the peaks of $\Delta V_C(t)$ come just before ("leads") or just after ("lags") the peak of $\Delta V_R(t)$? Show/explain your reasoning.

(ii) What is the phase difference between $\Delta V_R(t)$ and $\Delta V_L(t)$*?* Do the peaks of $\Delta V_R(t)$ come just before ("leads") or just after ("lags") the peak of $\Delta V_L(t)$? Show/explain your reasoning.

(d) Assuming for simplicity that $\delta = 0$, make a crude sketch on one graph of $\Delta V_R(t)$ and $\Delta V_C(t)$, being very careful to identify which is which. Does your sketch agree with your claim in part (c) about "leading" or "lagging"? (Briefly, comment)