Evolution of the Gaussian Wave Packet for the Free Particle

Today's homework is one "big problem" broken up into parts. We will be following all the steps to find $\Psi(x,t)$ for a free particle given an initially Gaussian wavefunction $\Psi(x,0)$ and then analyzing the properties of the result. This problem is broken into 5 parts.

In addition to completing this homework, the weekend will be good time for you to review all the things we've covered so far and try to organize your knowledge. What are the most important concepts and equations that we keep using? Write these in one convenient place for your reference while working on homework or the upcoming exam.

1) A Mathematical Digression

We've already used the simple Gaussian integral formula,

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \tag{1}$$

Using this, prove the more general expression, which will be useful later in this assignment,

$$\int_{-\infty}^{+\infty} e^{-ax^2-bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)} \tag{2}$$

To do this, do a "u" substitution by letting $y \equiv \sqrt{a}[x + (b/2a)]$ and write the argument for the exponential in terms of y; for reasons you will see, this is called "completing the square."

Particles can be described by approximately localized "lumps" of probability. Because they are in general built out of many stationary states, we think of them as a "packet" of plane waves and call such wave functions "wave packets." For example, consider a free particle ($V = 0$ everywhere) with the initial wave function

$$\Psi(x,0) = Ae^{-ax^2} \tag{3}$$

where a and A are constants, and a is real and positive.

2) Normalize $\boldsymbol{\Psi(x,0)}$ and find $\boldsymbol{\phi(k)}$

Normalize $\Psi(x,0)$ and calculate the Fourier transform distribution $\phi(k)$, which tells you how $\Psi(x,0)$ is built out of plane waves. The integral expression from the previous question will be useful here.

3) Find $\boldsymbol{\Psi(x,t)}$

Now calculate the wave function $\Psi(x,t)$ at all times. The integral expression is again useful. The answer will be:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+i\Omega t)}}{\sqrt{1+i\Omega t}} \tag{4}$$

where we have defined a new quantity, $\Omega \equiv 2a\hbar/m$.

4) Evolution of $|\Psi(x,t)|^2$ over time

Using the previous result, calculate the probability density $|\Psi(x,t)|^2$. Express your answer in terms of the quantity $w = \sqrt{a/(1+\Omega^2 t^2)}$. Sketch $|\Psi(x,t)|^2$ (as a function of x) at time $t = 0$ and then again at some much larger t. Be sure your sketches include labels of notable quantities or features on the vertical and horizontal axes. What happens to $|\Psi(x,t)|^2$ as time goes on?

5) Expectation values over time

You can find these as a function of w, which as you recall, is itself a function of t.

(a) Find $\langle x \rangle$ and $\langle p \rangle$.

(b) Find $\langle x^2 \rangle$.

(c) Using those two results and $\langle p^2 \rangle = a\hbar^2$, find σ_x and σ_p. Does the uncertainty principle hold? At what time *t* does the system come closest to the uncertainty limit?