1) Review of previous concepts

(a) For the ground state $\psi_0(x)$ of the harmonic oscillator, what is the probability of finding the particle *outside* of the classically allowed region? *Hint*: Classically, the energy of an oscillator is $E = \frac{1}{2}m\omega^2 a^2$, where a is the amplitude. So, the classically allowed region for an oscillator of energy E extends from $x = -\sqrt{2E/m\omega^2}$ to $x = +\sqrt{2E/m\omega^2}$. You'll need the help of a computer to do this integral, and you'll just need a decimal answer. It will be easiest to do if you define your integrand and limits of integration in terms of a new variable given in Equation 2.72, instead of as a function of x. For your final step, get to the point where $P = \frac{2}{\sqrt{\pi}}\int_1^\infty e^{-\xi^2}d\xi$, then use Wolfram Alpha or Mathematica. *Answer*: 0.157.

(b) Examine Figure 2.7 from Griffiths. Should the probability that you just calculated increase, decrease, or stay the same as n increases? Is this consistent with the correspondence principle; why or why not?

This is a nice application of some of our concepts from Chapter 1 (*using a probability density to find probabilities of finding a measurement in a certain range*), *and a comparison of classical and quantum examples.*

Section 2.5 *is a nice bridge between Sections* 2.4 *and* 2.6. *The solution to the Delta Potential Well provided us with an understanding of reflection and transmission coefficients, which we'll need to understand the Finite Square Well in* 2.6, *where the mathematics is more complicated. Section* 2.5 *also introduced* (*or re-introduced*) *use to the Delta function, which we'll use here to continue analyzing the concept of "momentum space" that we developed in Section* 2.4. *So, though this homework doesn't explicitly focus on the solution we developed in Section* 2.5, *it allows us to expand upon Section* 2.4 *and preview Section* 2.6.

2) Properties of Delta Functions

(a) Using Plancherel's theorem, show that we can write the Dirac Delta function as $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk$. *Hint*: Find the Fourier transform of $\delta(x)$.

(b) Define $\theta(x)$ to be a step function:

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Show that $\delta(x) = d\theta/dx$. *Hints*: Delta functions really only make sense under integral signs. Two expressions ($D_1(x)$ and $D_2(x)$) involving delta functions are equal if:

$$\int_{-\infty}^{+\infty} f(x)D_1(x)dx = \int_{-\infty}^{+\infty} f(x)D_2(x)dx \tag{1}$$

You will need to use integration by parts.

3) Momentum-Space Wave Functions

Although some people casually call $\phi(k)$ a "momentum distribution", we should be careful. Remembering that $p = \hbar k$, I claim the momentum distribution is really

$$\Phi(p) = \frac{1}{\hbar}\phi(k) \tag{2}$$

Convince me that $\Phi(p)$ is properly normalized (and consequently, why $1/\hbar$ is needed) by showing that

$$\int_{-\infty}^{+\infty} \Phi^*(p)\Phi(p)dp = 1 \tag{3}$$

You should assume only that $\int_{-\infty}^{+\infty} \Psi^*(x)\Psi(x)dx = 1$. You will need the expression from Problem 2a to simplify the triple integral you get when you expand $\phi(k)$ using Griffiths equation 2.104. Don't forget $dp = \hbar dk$.

4) Expectation Values in Momentum Space (continuation of the previous problem)

If you define $\hat{x} = i\hbar\frac{\partial}{\partial p}$, show that

$$\int_{-\infty}^{+\infty} \Phi^*(p)\hat{x}\Phi(p)dp = \langle x\rangle \tag{4}$$

You'll need the same expressions/substitutions as I listed at the end of the previous problem.

Notes on the meaning of this result: pay attention to the symmetry of what we're showing here. Before, we had:

$$\langle x\rangle = \int_{-\infty}^{+\infty} \Psi^*(x)\hat{x}\Psi(x)dx \text{ where } \hat{x} = x \tag{5}$$

$$\langle p\rangle = \int_{-\infty}^{+\infty} \Psi^*(x)\hat{p}\Psi(x)dx \text{ where } \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{6}$$

Now we have:

$$\langle x\rangle = \int_{-\infty}^{+\infty} \Phi^*(p)\hat{x}\Phi(p)dp \text{ where } \hat{x} = i\hbar\frac{\partial}{\partial p} \tag{7}$$

It looks like when we swap $\Psi(x)$ with $\Phi(p)$, we also swap operators, so by symmetry, we will have:

$$\langle p\rangle = \int_{-\infty}^{+\infty} \Phi^*(p)\hat{p}\Phi(p)dp \text{ where } \hat{p} = p \tag{8}$$

$\Phi(p)$ plays the same role as $\Psi(x)$ does. It normalizes like $\Psi(x)$, and it contains information about the system (expectation values, etc.) just like $\Psi(x)$. Since it is written as a function of p instead of x, it is called a momentum-space wave function, with $|\Phi(p)|^2$ being the momentum-space probability density ($\int_{p_a}^{p_b}|\Phi(p)|^2dp$ gives the probability of finding the particle with a momentum p between p_a and p_b).