1) Vector Spaces

(a) Consider ordinary 3D vectors $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ (with complex components) together with the set of scalars. For each of the following three cases, find out whether it constitutes a vector space. If so, what is its dimension? If not, why not?

(i) The subset of all vectors with $A_z = 0$.
(ii) The subset of all vectors with $A_z = 1$.
(iii) The subset of all vectors whose components are all equal.

(b) Does the set of all 2 × 2 matrices form a vector space? Assume the usual rules for matrix addition and multiplication by a scalar:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

If it does not form a vector space, why not? If it does, state the dimensionality and give an example of a set of basis vectors.

(c) Does the set of all functions $f(x)$ defined on the range $0 < x < 1$ that vanish at $x = 0$ and $x = 1$ together with the set of scalars form a vector space? If not, why not? If it does, state the dimensionality, and try to think of a set of possible basis vectors.

You should definitely review Appendix A for this problem. In it you'll find the definition of a vector space, and two more properties of vector spaces (*there exists a null vector, and every vector has an inverse vector*). *The dimensionality of a vector space is the number of vectors in the basis set. Ahh why am I writing this all out, go check out Appendix A*!

2) Hermitian Operators

(a) Show that the sum of two Hermitian operators is a Hermitian operator.

(b) Suppose that $\hat{A}$ is a Hermitian operator and α is a number. Under what condition on α is $\alpha\hat{A}$ Hermitian operator?

(c) When is the product of two Hermitian operators a Hermitian operator?

On this problem, use our new notation, $\left\langle f \middle| \hat{Q} g \right\rangle$. *You don't need to explicitly write out any integrals! Each part takes only a few lines; work smarter, not harder.*

3) Commutators

Recall that the definition of the commutator of $\hat{A}$ and $\hat{B}$ is $\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ (Eq. 2.49). We showed in class that $\hat{x}$ and $\hat{p}$ do not commute. More specifically, $[\hat{x}, \hat{p}] = i\hbar$ (Eq. 2.52). Recall also that the definition of the Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$.

(a) What is the commutator of $\hat{x}$ and $\hat{H}$? Are there any potentials $V(x)$ in which $\hat{x}$ and $\hat{H}$ commute? (Answer to first part: $\left[\hat{x}, \hat{H}\right] = \frac{i\hbar\hat{p}}{m}$)

(b) What is the commutator of $\hat{p}$ and $\hat{H}$? Are there any potentials $V(x)$ in which $\hat{p}$ and $\hat{H}$ commute? (Answers: $\left[\hat{p}, \hat{H}\right] = \frac{\hbar}{i}\frac{\partial V(x)}{\partial x}$. Yes, for the free particle; be sure you can explain why.)

You should complete this problem before the next one. You'll need to use the result on the worksheet. Moreover, we'll be returning to commutators again in Section 3.5*, so this is a good preview for the next class.*

4) Position, Momentum, and Energy Measurements

Complete the 4 pages titled “Homework: Position, Momentum, and Energy Measurements” that follow this page. Read each question carefully. I suggest marking up the PDF, but you could also print, handwrite, and scan if you want.

HOMEWORK: POSITION, MOMENTUM, AND ENERGY MEASUREMENTS

Name ____________________

1. The following questions focus on the momentum operator in quantum mechanics.

   a. Write an equation (in terms of x and derivatives with respect to x) relaying an eigenfunction $k_i(x)$ of the *momentum* operator $\hat{p}$ and its corresponding eigenvalue λ_i. Indicate explicitly which symbols are functions of x and which are not. Explain.

   b. Solve this equation for the eigenfunction corresponding to the eigenvalue λ_i, and sketch a graph of at least one example. Show your work.

   c. Are there any values of λ_i for which there are no (non-trivial) solutions to the equation above? Use your answer to determine all the allowed eigenvalues of the momentum operator. Explain.

   d. Consider the operators for position, $\hat{x}$, momentum, $\hat{p}$, and energy, $\hat{H}$. Determine which pairs of operators commute with each other. If any of your results depend on the system under consideration, state so explicitly. Explain.

   e. Consider the eigenfunctions of position, $\hat{x}$, momentum, $\hat{p}$, and energy, $\hat{H}$. Are the eigenfunctions of these three operators *the same* or *different*. If your answer depends on the system, state so explicitly. Explain.

   What can you say about the eigenfunctions of two operators that do not commute? Explain.

	Table 1: Number of Possible Measurement Outcomes (at a later time)			Table 2: Time-dependence of Probabilities of Possible Outcomes		
	Energy Eigenstate	Position Eigenstate	Momentum Eigenstate	Energy Eigenstate	Position Eigenstate	Momentum Eigenstate
Energy Measurements						
Position Measurements						
Momentum Measurements						

f. Use your results from the tutorial to fill in the portions of the table above that correspond to position and energy measurements. Briefly explain how you were able to determine these answers.

g. Complete the remaining portions of the table for momentum measurements and eigenstates of momentum. If your results depend on the system, state so explicitly. Explain how you determined your answers. (*Hint:* Consider your answers to the questions on the previous page.)

h. Consider a hypothetical operator, $\hat{O}$, that commutes with the Hamiltonian.

   i. If you started in an eigenstate of $\hat{O}$, which observables would have multiple possible results? Explain.

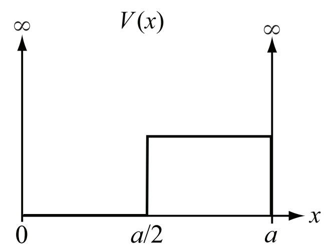
   ii. If you started in an eigenstate of $\hat{O}$, which observables would have probabilities that depend on time? Explain.

2. Consider a particle that is initially in the ground state of the infinite square well (with energy eigenvalues E_n). The initial wave function for this particle is given by $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 < x < a$.

   a. At $t = 0$, is the probability that the particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

   b. At a later time, $t_1 > 0$, is the probability that the particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

   c. Suppose you were to measure the energy of this particle at time t_1 (assume no previous measurements have been made). What is the probability that the result of this measurement will be E_1? Explain.

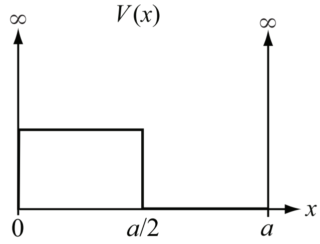


Consider a particle in the potential shown at right, which consists of a step added to the right half of an infinite square well. The initial wave function of this particle is the same as for the particle above: $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 < x < a$.

   d. At $t = 0$, is the probability that this particle is found on the left half of the well *greater than, less than,* or *equal to* the probability it is found on the right half of the well? Explain.

   e. Suppose you were to measure the energy of this particle at time t_1 (assume no other measurements have been made). What is the probability that the result of this measurement will be E_1? Explain.

Now consider a particle in the potential at right, which consists of a step of the same height added to the *left* half of an infinite square well. The wave function of this particle is the same as the particles above: $\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$ for $0 < x < a$.



f. At $t = 0$, is the probability of measuring the position to be near $x = a/4$ in this system *greater than, less than,* or *equal to* the probability of measuring the position to be near $x = a/4$ in the previous system (with the step on the right half of the well)? Explain.

g. At $t = 0$, is the probability of measuring the momentum to be to the left with magnitude $p = 2\pi\hbar/a$ in the system with the new potential *greater than, less than,* or *equal to* the probability of measuring the momentum to be to the left with magnitude $p = 2\pi\hbar/a$ in the system with the previous potential? Explain.

h. Consider the following *incorrect* statement.

"The potential for a system tells me everything I need to know about it. It tells me the allowed energies; the functional form and the time dependence of the wave function; and the probability of each possible outcome of a measurement of energy, position, or momentum."

Identify which portions of the statement are incorrect. Explain your reasoning.