For most of these problems, I take a lot more ink on the page to ask the question than it will take you to provide the answer. Don't be intimidated by the apparent length of this assignment; it looks so long because I provide you with so much information (*rather than asking you to derive it all on your own*).

1) Momentum-Space Wave Function

Find the momentum-space wave function, $\Phi(p, t)$, for a particle in the ground state of the harmonic oscillator. What is the probability (to 2 significant digits) that a measurement of p on a particle in this state would yield a value outside the classical range (for the same energy)?

Hints:

- When finding $\Phi(p, t)$, use the "completing the square" trick that you derived in the Section 2.4 homework.
- Remember that classically, the maximum momentum (positive or negative) would be achieved at equilibrium, when total energy is equal to kinetic energy $\frac{p^2}{2m} = E$. Express $|p_{max}|$ as a function of m and ω.
- The probability of being outside the classically allowed region is equal to (1 – probability of being inside the allowed region).
- We asked a similar question (probability of finding *position* outside of classically allowed range of harmonic oscillator) in Section 2.3. Like then, you'll need Mathematica or Wolfram Alpha to numerically calculate your final integral for probability.

Here we are using the generalized statistical interpretation (*Section* 3.4) *to use the momentum-space wavefunction to tell us about probabilities of momentum measurements. We're conceptually doing something similar to what we've been doing in "position-space" with* $\Psi(x, t)$.

2) Quantum Measurements

An operator $\hat{A}$ (representing observable A) has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2. Operator $\widehat{B}$ (representing observable B) has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. Suppose these eigenstates are related by the following:

$$\psi_1 = \frac{2}{3}\phi_1 + \frac{\sqrt{5}}{3}\phi_2 \quad \text{and} \quad \psi_2 = -\frac{\sqrt{5}}{3}\phi_1 + \frac{2}{3}\phi_2 \tag{1}$$

(An analogy to the above is how we can write position eigenstates as an infinite sum of energy eigenstates, or energy eigenstates as an infinite sum of position eigenstates.)

(a) Show me that (assuming ϕ_1 and ϕ_2 are properly normalized) ψ_1 and ψ_2 are normalized. *Use inner product notation, no need to write integrals*.

(b) Let's start in some unspecified random state. You then measure observable A. Further, assume that you do in fact measure the particular value a_1. What is the state of the system immediately after this measurement?

(c) Immediately after the measurement of A (which, recall, happened to yield a_1), the observable B is then measured. What are the possible results of the B measurement, and what are their probabilities?

(d) Consider the three following scenarios, following up on the above measurements.

i. Suppose I tell you that, immediately after measuring B, we found b_1. And immediately after that, we measure A again. What is the probability of getting outcome a_1 again? *Hint*: *Express* ϕ_1 and ϕ_2 *as superpositions of* ψ_1 and ψ_2.

ii. Suppose instead that you measure B, but you do not know the outcome! (So there are two possibilities, b_1 or b_2, with relative probabilities given in part (c) above.) If you NOW measure A, what is the probability of getting a_1? Hint: How do you deal with probabilities when you have "or" and "and" scenarios?

iii. Go back a step — suppose that after our very first measurement of A (when, you will recall, we happened to find a_1) that we had completely neglected to measure B at all, and simply measured A again, right away. What would be the possible result(s) of the second measurement of A, with what probabilities?

Briefly, discuss all three answers to part (d); try to make sense of all this! (Do you think the operators $\hat{A}$ and $\widehat{B}$ commute? Why/why not? Is there any classical analogue to this, or is it pure quantum weirdness?)

The remaining problems give us practice representing 2- *or* 3-*level systems as vectors, with operators as matrices. We can still apply the same questions we've been asking to these n-level systems, e.g., what is the expectation value of an observable? Are two observables compatible? What is the time-dependence any arbitrary state, if I know how to express it in terms of the stationary states (energy eigenfunctions) and corresponding energy eigenvalues*?

Griffiths Appendix A.5 *may be helpful for reminding you how to find eigenvalues and eigenvectors, but I'd prefer you focus on the concepts instead of doing the linear algebra grunt work, so please use a computer. There is only one* (*non-trivial*) *matrix for which I ask you to find all eigenvectors and eigenvalues, which you may do using Mathematica. Note Mathematica will not give normalized eigenvectors. You may find these two references helpful*:

https://reference.wolfram.com/language/howto/CreateAMatrix.html
https://reference.wolfram.com/language/tutorial/EigenvaluesAndEigenvectors.html

3) **3-Level Hamiltonian**

The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \tag{2}$$

where a, b, and c are real numbers. Assume $a - c \neq \pm b$.

(a) Find the eigenvalues (E_1, E_2, and E_3) and (normalized) eigenvectors ($|s_1\rangle$, $|s_2\rangle$, and $|s_3\rangle$) of the Hamiltonian. *See note above about Mathematica.*

(b) If the system starts out in the state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{3}$$

what is $|\mathcal{S}(t)\rangle$? Is this a stationary state?

(b) If the system starts out in the state

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

what is $|\mathcal{S}(t)\rangle$? Is this a stationary state?

4) Observables Part I

Two observables, A and B, are represented by the matrices:

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

where λ and μ are positive real numbers.

Imagine also that the Hamiltonian for this three-level system is represented by the matrix

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{6}$$

where ω is also a positive real number.

(a) Do **A** and **B** commute? Explain what your answer tells you about A and B.

(b) Find the eigenvalues and (normalized) eigenvectors of **H**. *Hint*: *It's a diagonal matrix*!

(c) Is the spectrum of this Hamiltonian degenerate?

5) Observables Part II

Use the same matrices from the previous question, and this new information:

The eigenvalues for **A** are 2λ, λ, and $-\lambda$ with corresponding eigenvectors:

$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad |a_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |a_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{7}$$

The eigenvalues for **B** are 2μ, μ, and $-\mu$ with corresponding eigenvectors:

$$|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |b_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad |b_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{8}$$

Now suppose the system starts out in the generic state:

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{9}$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$

(a) Find the expectation values (at time $t = 0$) of H, A, and B.

(b) What is $|\mathcal{S}(t)\rangle$?

(c) If you measured the energy of this state (at time t), what values might you get, and what is the probability of each? Do the probabilities depend on time?

(d) If you measured the observable A of this state (at time t), what values might you get, and what is the probability of each? Do the probabilities depend on time?

(e) Confirm that the sum of the probabilities you found in part (d) add up to 1.

5) Operators as Matrices

Print and complete the two-page worksheet below titled "Homework: Dirac Notation". (Ignore the fact that it seems to start at question 4.)

This one gives us a visual feel for what operators "do" and what commuting or not means!

HOMEWORK: DIRAC NOTATION

Name ____________________

4. Consider the two-dimensional spatial vectors written at right in Dirac notation.

$$|u\rangle = -3|x\rangle + 3|y\rangle$$
$$|v\rangle = 3|x\rangle - |y\rangle$$

a. Are these vectors normalized? If not, calculate the normalized vectors, $|u'\rangle$ and $|v'\rangle$. What important property does a normalized vector have?

b. Represent the normalized $|u'\rangle$ and $|v'\rangle$ as column vectors. Express each element as both an inner product and a number

c. Consider the operator $\hat{P}$ defined by the matrix at right.

$$\hat{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

i. Graph $|u'\rangle$ and $\hat{P}|u'\rangle$ on the same set of axes in the space below.

ii. Graph $|v'\rangle$ and $\hat{P}|v'\rangle$ on the same set of axes in the space below.

iii. Determine a general rule for the action of the operator $\hat{P}$. (*Hint:* Consider the graphical result of the operator's action on each vector.)

d. Now consider the operator $\hat{Q}$ defined by the matrix at right.

$$\hat{Q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

i. Graph $|u'\rangle$ and $\hat{Q}|u'\rangle$ on the same set of axes in the space below.

ii. Graph $|v'\rangle$ and $\hat{Q}|v'\rangle$ on the same set of axes in the space below.

iii. Determine a general rule for the action of the operator $\hat{Q}$.

e. Suppose you were to act both operators $\hat{P}$ and $\hat{Q}$ on one of the vectors above. Would the result depend on the order in which you chose to act the two operators? Explain using both the mathematical result and the graphical interpretation of each operator.