Game Plan for Solution of the TISE of the SHO

Step 1:	Clean up TISE using intermediate quantities	$\psi_E(x) \rightarrow \psi_K(\xi)$
Step 2:	Find the solution in the asymptotic limit $\xi \rightarrow \pm\infty$	$\psi_K(\xi) \rightarrow \chi_{as}(\xi)$
Step 3:	Factor out the asymptotic behavior	$\psi_K(\xi) \rightarrow A h_K(\xi)\chi_{as}(\xi)$
Step 4:	Derive a DE for the unknown factor	equation for $h_K(\xi)$
Step 5:	Expand the unknown function in a power series	$h_K(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$
Step 6:	Put series in DE & derive recurrence relation	a_j in terms of a_{j-1}
Step 7:	Enforce boundary conditions	quantize $K_n \rightarrow E_n$

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$$\left(\frac{d^2}{dx^2} - \frac{m^2\omega^2}{\hbar^2}x^2 + \frac{2mE}{\hbar^2}\right)\psi_E(x) = 0$$

TISE for SHO

Quantity	Definition	Units
energy	$K \equiv \frac{2E}{\hbar\omega}$	dimensionless
potential strength	$\alpha \equiv \sqrt{\frac{m\omega}{\hbar}}$	$(\text{length})^{-1}$
position	$\xi \equiv \alpha x$	dimensionless

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \alpha\frac{d\psi}{d\xi}$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \frac{d}{d\xi}\frac{d\xi}{dx}\left(\frac{d\psi}{dx}\right) = \alpha^2\frac{d^2\psi}{d\xi^2}$$

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + K\right)\psi_K(\xi) = 0$$

Weber's equation

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Asymptotic solution (singularity hunt), $\xi \to \pm\infty$

$$\psi_K(\xi) \xrightarrow[\xi\to\pm\infty]{} \chi_{as}(\xi) \quad \Rightarrow \quad \left(\frac{d^2}{d\xi^2} - \xi^2\right)\chi_{as}(\xi) = 0$$

$$\chi_{as}(\xi) = Ae^{-\xi^2/2} + Be^{+\xi^2/2} = Ae^{-\xi^2/2}$$

Factor out asymptotic behavior

$$\psi_K(\xi) = A\,h_K(\xi)\,e^{-\xi^2/2}$$

$$\frac{d\psi}{d\xi} = A\left(\frac{dh}{d\xi} - \xi h\right)e^{-\xi^2/2}$$

$$\frac{d^2\psi}{d\xi^2} = \frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right) = \left(\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + \left(\xi^2 - 1\right)h\right)e^{-\xi^2/2}$$

$$\left(\frac{d^2}{d\xi^2} - 2\xi\frac{d}{d\xi} + (K-1)\right)h_K(\xi) = 0$$

Hermite equation