1.4 DESCRIBING THE 3D WORLD: VECTORS

Physical phenomena take place in the 3D world around us. In order to be able to make quantitative predictions and give detailed, quantitative explanations, we need tools for describing precisely the positions and velocities of objects in 3D, and the changes in position and velocity due to interactions. These tools are mathematical entities called 3D "vectors." A symbol denoting a vector is written with an arrow over it:

$$\vec{r} \text{ is a vector}$$

In three dimensions a vector is a triple of numbers $\langle x, y, z \rangle$. Quantities like the position or velocity of an object can be represented as vectors:

$$\vec{r}_1 = \langle 3.2, -9.2, 66.3 \rangle \text{ m} \quad \text{(a position vector)}$$
$$\vec{v}_1 = \langle -22.3, 0.4, -19.5 \rangle \text{ m/s} \quad \text{(a velocity vector)}$$

Many vectors have units associated with them, such as meters or meters per second. In this course, we will work with the following important physical quantities that are vectors: position, velocity, rate of change of velocity (acceleration), momentum, rate of change of momentum, force, angular momentum, torque, electric field, magnetic field, energy flow, and momentum flow. All of these vectors have associated physical units.

We use the notation $\langle x, y, z\rangle$ for vectors because it emphasizes the fact that a vector is a single entity, and because it is easy to work with. This notation appears in many calculus textbooks; you will probably encounter other ways of expressing vectors mathematically as well.

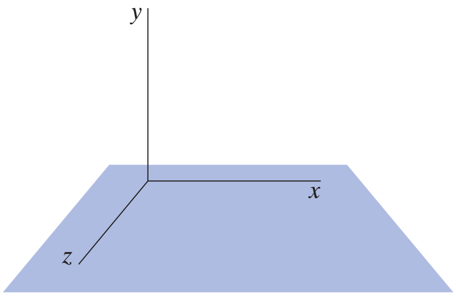

Figure 1.11 Right-handed 3D coordinate system. The xy plane is in the plane of the page, and the z axis projects out of the page, toward you.

Position Vectors

A position vector is a simple example of a physical vector quantity. We will use a 3D Cartesian coordinate system to specify positions in space and other vector quantities. Usually we will orient the axes of the coordinate system as shown in Figure 1.11: $+x$ axis to the right, $+y$ axis upward, and $+z$ axis coming out of the page, toward you. This is a "right-handed" coordinate system: if you hold the thumb, first, and second fingers of your right hand perpendicular to each other, and align your thumb with the x axis and your first finger with the y axis, your second finger points along the z axis. In some math textbook discussions of 3D coordinate systems, the x axis points out, the y axis points to the right, and the z axis points up. This is the same right-handed coordinate system, viewed from a different "camera position." Since we will sometimes consider motion in a single plane, it makes sense to orient the xy plane in the plane of a vertical page or computer display, so we will use the viewpoint in which the y axis points up.

A position in 3D space can be considered to be a vector, called a *position vector*, pointing from an origin to that location. Figure 1.12 shows a position vector, represented by an arrow with its tail at the origin, that might represent your final position if you started at the origin and walked 4 meters along the x axis, then 2 meters parallel to the z axis, then climbed a ladder so you were 3 meters above the ground. Your new position relative to the origin is a vector that can be written like this:

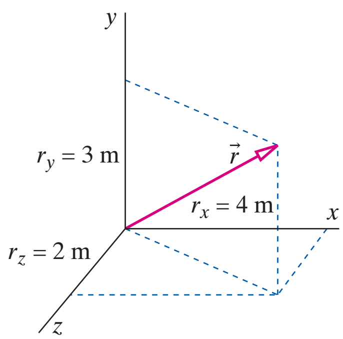

Figure 1.12 A position vector $\vec{r} = \langle 4, 3, 2\rangle$ m and its x, y, and z components.

$$\vec{r} = \langle 4, 3, 2\rangle \text{ m}$$

Each of the numbers in the triple is called a "component" of the vector, and is associated with a particular axis. Usually the components of a vector are denoted symbolically by the subscripts x, y, and z:

$$\vec{v} = \langle v_x, v_y, v_z\rangle \quad \text{(a velocity vector)}$$
$$\vec{r} = \langle r_x, r_y, r_z\rangle \quad \text{(a position vector)}$$
$$\vec{r} = \langle x, y, z\rangle \quad \text{(alternative notation for a position vector)}$$

The components of the position vector $\vec{r} = \langle 4, 3, 2\rangle$ m are:

$$r_x = 4\text{ m} \quad \text{(the } x \text{ component)}$$
$$r_y = 3\text{ m} \quad \text{(the } y \text{ component)}$$
$$r_z = 2\text{ m} \quad \text{(the } z \text{ component)}$$

The x component of the vector $\vec{v}$ is the number v_x. The z component of the vector $\vec{v}_1 = \langle -22.3, 0.4, -19.5\rangle$ m/s is -19.5 m/s. A component such as v_x is not a vector, since it is only one number.

QUESTION Can a vector be zero?

The zero vector $\langle 0,0,0 \rangle$ is a legal vector, which we will sometimes write as $\vec{0}$. A zero position vector describes the position of an object located at the origin. A zero velocity vector describes the velocity of an object that is at rest at a particular instant.

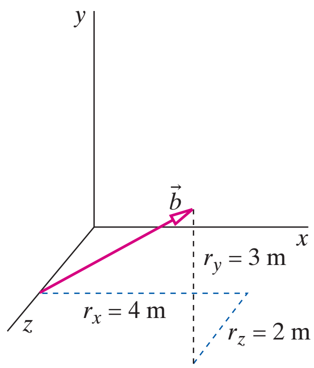

Figure 1.13 The arrow represents the vector $\vec{b} = \langle 4,3,2 \rangle$ m, drawn with its tail at location $\langle 0,0,2 \rangle$.

Drawing Vectors

A position vector is special in that its tail is always at the origin of a coordinate system, but this is not the case for other vectors. It is important to note that the x component of a vector specifies the difference between the x coordinate of the tail of the vector and the x coordinate of the tip of the vector. It does not give any information about the location of the tail of the vector (compare Figures 1.12 and 1.13). By convention, arrows representing vector quantities such as velocity are usually drawn with the tail of the arrow at the location of the object.

In Figure 1.12 we represented your position vector relative to the origin graphically by an arrow whose tail is at the origin and whose arrowhead is at your position. The length of the arrow represents the distance from the origin, and the direction of the arrow represents the direction of the vector, which is the direction of a direct path from the initial position to the final position (the "displacement"; by walking and climbing you "displaced" yourself from the origin to your final position).

Since it is difficult to draw a 3D diagram on paper, when working on paper you will usually be asked to draw vectors that all lie in a single plane. Figure 1.14 shows an arrow in the xy plane representing the vector $\langle -3,-1,0 \rangle$.

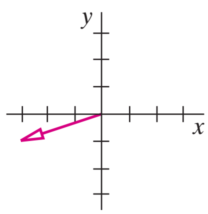

Figure 1.14 The position vector $\langle -3,-1,0 \rangle$, drawn at the origin, in the xy plane. The components of the vector specify the displacement from the tail to the tip. The z axis, which is not shown, comes out of the page, toward you.

Scalars

A quantity that is represented by a single number is called a *scalar*. A scalar quantity does not have a direction. Examples include the mass of an object, such as 50 kg, or the temperature, such as $-20\,^\circ$C. Vectors and scalars are very different entities; a vector can never be equal to a scalar, and a scalar cannot be added to a vector. Scalars can be positive, negative, or zero:

$$m = 50\,\text{kg}$$
$$T = -20\,^\circ\text{C}$$

Vector Operations

Vectors are mathematical entities, and have their own mathematical operations. Some of these operations are the same as those you already know for scalars. Others, such as multiplication, are quite different, and division by a vector is not legal. Here are the vector operations that we will discuss and use in this textbook:

VECTOR OPERATIONS

Mathematical operations that are defined for vectors:

- Multiply or divide a vector by a scalar: $2\vec{a}$, $\vec{v}/5$
- Find the magnitude of a vector: $|\vec{a}|$
- Find a unit vector giving direction: $\hat{a}$
- Add one vector to another: $\vec{a}+\vec{b}$
- Subtract one vector from another: $\vec{a}-\vec{b}$
- Differentiate a vector: $d\vec{r}/dt$
- Dot product of two vectors (result is a scalar): $\vec{a} \bullet \vec{b}$
- Cross product of two vectors (result is a vector): $\vec{a} \times \vec{b}$

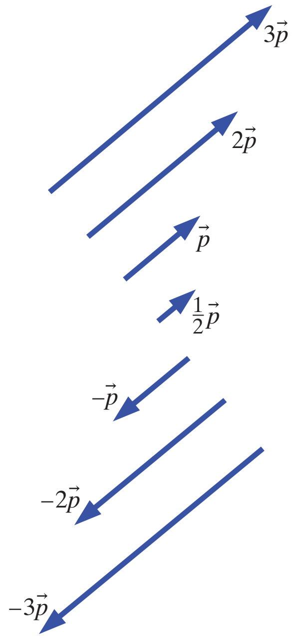

Figure 1.15 Multiplying a vector by a scalar changes the magnitude of the vector. Multiplying by a negative scalar reverses the direction of the vector.

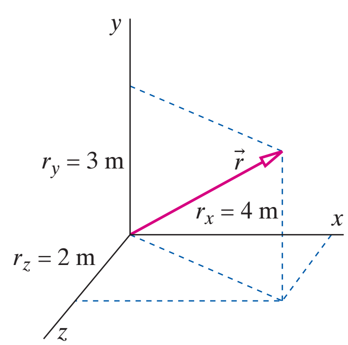

Figure 1.16 A vector representing a displacement from the origin.

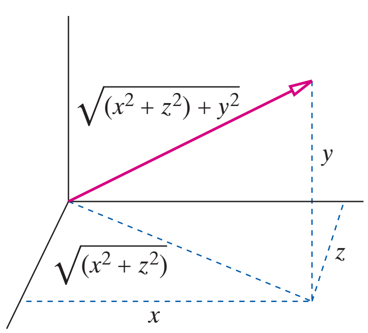

Figure 1.17 The magnitude of a vector is the square root of the sum of the squares of its components (3D version of the Pythagorean theorem).

The dot product will be introduced in Chapter 5, and the cross product in Chapter 11.

There are certain operations that are neither legal nor meaningful for vectors:

- A vector cannot be set equal to a scalar.
- A vector cannot be added to or subtracted from a scalar.
- A vector cannot occur in the denominator of an expression. (Although you can't divide by a vector, note that you can legally divide by the *magnitude* of a vector, which is a scalar.)
- As with scalars, you can't add or subtract vectors that have different units.

Multiplying a Vector by a Scalar

A vector can be multiplied (or divided) by a scalar. If a vector is multiplied by a scalar, each of the components of the vector is multiplied by the scalar:

$$\text{If } \vec{r} = \langle x, y, z \rangle, \text{ then } a\vec{r} = \langle ax, ay, az \rangle$$

$$\text{If } \vec{v} = \langle v_x, v_y, v_z \rangle, \text{ then } \frac{\vec{v}}{b} = \left\langle \frac{v_x}{b}, \frac{v_y}{b}, \frac{v_z}{b} \right\rangle$$

$$\frac{1}{2}\langle 6, -20, 9 \rangle = \langle 3, -10, 4.5 \rangle$$

Multiplication by a scalar "scales" a vector, keeping its direction the same but making its magnitude larger or smaller (Figure 1.15). Multiplying by a negative scalar reverses the direction of a vector.

$$(-1)\langle 0, 0, 4 \rangle = \langle 0, 0, -4 \rangle$$

Checkpoint 3 You stand at location $\vec{r} = \langle 2, -3, 5 \rangle$ m. Your friend stands at location $\vec{r}/2$. What is your friend's position vector?

Magnitude

Figure 1.16 shows a vector representing a displacement of $\langle 4, 3, 2 \rangle$ m from the origin. What is the distance from the tip of this vector to the origin? Using a 3D extension of the Pythagorean theorem for right triangles (Figure 1.17), we find that

$$\sqrt{(4\,\text{m})^2 + (3\,\text{m})^2 + (2\,\text{m})^2} = \sqrt{29}\,\text{m} = 5.39\,\text{m}$$

We say that the *magnitude* $|\vec{r}|$ of the position vector $\vec{r}$ is

$$|\vec{r}| = 5.39\,\text{m}$$

The magnitude of a vector is written either with absolute-value bars around the vector as $|\vec{r}|$, or simply by writing the symbol for the vector without the little arrow above it, r.

MAGNITUDE OF A VECTOR

$$\text{If the vector } \vec{r} = \langle r_x, r_y, r_z \rangle \text{ then } |\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} \text{ (a scalar).}$$

The magnitude of a vector is always a positive number. The magnitude of a vector is a single number, not a triple of numbers, and it is a scalar, not a vector.

You may wonder how to find the magnitude of a quantity like $-3\vec{r}$, which involves the product of a scalar and a vector. This expression can be factored:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}|$$

The magnitude of a scalar is its absolute value, so:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}| = 3\sqrt{r_x^2 + r_y^2 + r_z^2}$$

Checkpoint 4 If $\vec{v} = \langle 2, -3, 5 \rangle$ m/s, what is $\left|-\frac{1}{2}\vec{v}\right|$?

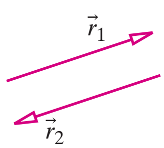

Figure 1.20 Are these two vectors equal? (Checkpoint 5)

EQUALITY OF VECTORS

A vector is equal to another vector if and only if all the components of the vectors are equal.

$$\vec{w} = \vec{r} \quad \text{means that}$$
$$w_x = r_x \quad \text{and} \quad w_y = r_y \quad \text{and} \quad w_z = r_z$$

The magnitudes and directions of two equal vectors are the same:

$$|\vec{w}| = |\vec{r}| \quad \text{and} \quad \hat{w} = \hat{r}$$

Checkpoint 5 (a) Consider the vectors $\vec{r}_1$ and $\vec{r}_2$ represented by arrows in Figure 1.20. Are these two vectors equal? **(b)** If $\vec{a} = \langle 400, 200, -100 \rangle$ m/s^2, and $\vec{c} = \vec{a}$, what is the unit vector $\hat{c}$ in the direction of $\vec{c}$?

Vector Addition and Subtraction

Vectors may be added, and one vector may be subtracted from another vector. However, a scalar cannot be added to or subtracted from a vector.

ADDING AND SUBTRACTING VECTORS

The sum or difference of two vectors is another vector, obtained by adding or subtracting the components of the vectors. Given two vectors $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$, then

$$\vec{A} + \vec{B} = \langle (A_x + B_x), (A_y + B_y), (A_z + B_z) \rangle$$
$$\langle 1, 2, 3 \rangle + \langle -4, 5, 6 \rangle = \langle -3, 7, 9 \rangle$$
$$\vec{A} - \vec{B} = \langle (A_x - B_x), (A_y - B_y), (A_z - B_z) \rangle$$
$$\langle 1, 2, 3 \rangle - \langle -4, 5, 6 \rangle = \langle 5, -3, -3 \rangle$$

If $\vec{C} = \vec{A} + \vec{B}$, then $\vec{C} - \vec{B} = \vec{A}$ and so on, just as in scalar addition and subtraction. Note also that $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$, which is sometimes useful in the context of graphical subtraction (see below).

QUESTION Is adding the magnitudes of two vectors equivalent to adding two vectors, then taking the magnitude?

No. The magnitude of a vector is *not* in general equal to the sum of the magnitudes of the two original vectors! For example, the magnitude of the vector $\langle 3,0,0\rangle$ is 3, and the magnitude of the vector $\langle -2,0,0\rangle$ is 2, but the magnitude of the vector $(\langle 3,0,0\rangle + \langle -2,0,0\rangle)$ is 1, not 5!

Checkpoint 6 If $\vec{F}_1 = \langle 300, 0, -200\rangle$ and $\vec{F}_2 = \langle 150, -300, 0\rangle$, calculate the following quantities and make the requested comparisons: **(a)** $\vec{F}_1 + \vec{F}_2$ **(b)** $|\vec{F}_1 + \vec{F}_2|$ **(c)** $|\vec{F}_1| + |\vec{F}_2|$ **(d)** Is $|\vec{F}_1 + \vec{F}_2| = |\vec{F}_1| + |\vec{F}_2|$? **(e)** $\vec{F}_1 - \vec{F}_2$ **(f)** $|\vec{F}_1 - \vec{F}_2|$ **(g)** $|\vec{F}_1| - |\vec{F}_2|$ **(h)** Is $|\vec{F}_1 - \vec{F}_2| = |\vec{F}_1| - |\vec{F}_2|$?

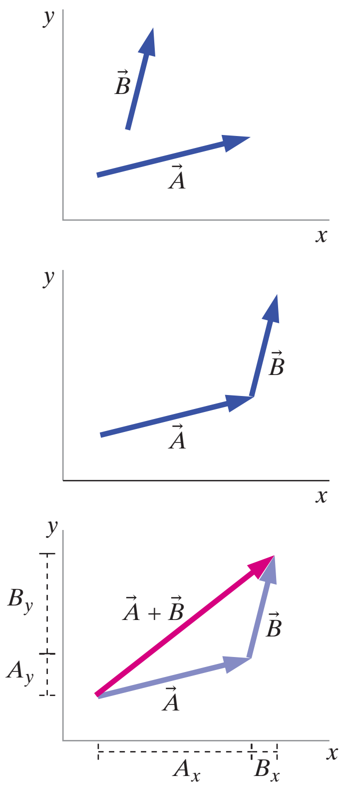

Figure 1.21 The procedure for adding two vectors graphically: To add $\vec{A} + \vec{B}$ graphically, move $\vec{B}$ so the tail of $\vec{B}$ is at the tip of $\vec{A}$, then draw a new arrow starting at the tail of $\vec{A}$ and ending at the tip of $\vec{B}$.

The sum of two vectors has a geometric interpretation. In Figure 1.21 you first walk along displacement vector $\vec{A}$, followed by walking along displacement vector $\vec{B}$. What is your net displacement vector $\vec{C} = \vec{A} + \vec{B}$? The x component C_x of your net displacement is the sum of A_x and B_x. Similarly, the y component C_y of your net displacement is the sum of A_y and B_y.

GRAPHICAL ADDITION OF VECTORS

To add two vectors $\vec{A}$ and $\vec{B}$ graphically (Figure 1.21):

- Draw the first vector $\vec{A}$.
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tip* of the first vector.
- Draw a new vector from the tail of vector $\vec{A}$ to the tip of vector $\vec{B}$.

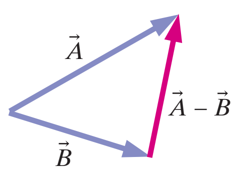

Figure 1.22 The procedure for subtracting vectors graphically: Draw vectors tail to tail; draw a new vector from the tip of the second vector to the tip of the first vector.

GRAPHICAL SUBTRACTION OF VECTORS

To subtract one vector $\vec{B}$ from another vector $\vec{A}$ graphically (Figure 1.22):

- Draw the first vector $\vec{A}$.
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tail* of the first vector.
- Draw a new vector from the tip of vector $\vec{B}$ to the tip of vector $\vec{A}$.

Note that you can check this algebraically and graphically. As shown in Figure 1.22, since the tail of $\vec{A} - \vec{B}$ is located at the tip of $\vec{B}$, then the vector $\vec{A}$ should be the sum of $\vec{B}$ and $\vec{A} - \vec{B}$, as indeed it is:

$$\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$$

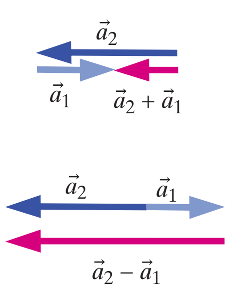

Figure 1.23 To add (top diagram) and subtract (bottom diagram) collinear vectors graphically, we offset the arrows slightly for clarity.

Graphical addition and subtraction of collinear vectors would be messy and difficult to interpret if we actually drew the arrows on top of each other. To make diagrams easier to interpret, we typically offset arrows slightly so we can see the results (Figure 1.23).

Checkpoint 7 Which of the following statements about the three vectors in Figure 1.24 are correct?
(a) $\vec{s} = \vec{t} - \vec{r}$ **(b)** $\vec{r} = \vec{t} - \vec{s}$ **(c)** $\vec{r} + \vec{t} = \vec{s}$ **(d)** $\vec{s} + \vec{t} = \vec{r}$ **(e)** $\vec{r} + \vec{s} = \vec{t}$

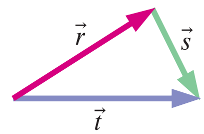

Figure 1.24 Checkpoint 7.

Commutativity and Associativity

Vector addition is commutative:

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

Vector subtraction is *not* commutative:

$$\vec{A}-\vec{B}\neq\vec{B}-\vec{A}$$

The associative property holds for vector addition and subtraction:

$$(\vec{A}+\vec{B})-\vec{C}=\vec{A}+(\vec{B}-\vec{C})$$

Applications of Vector Subtraction

Since we are interested in changes caused by interactions, we will frequently need to calculate the change in a vector quantity. For example, we may want to know the change in a moving object's position or the change in its velocity during some time interval. Finding such changes requires vector subtraction.

The Greek letter Δ (capital delta suggesting "D for Difference") is traditionally used to denote the change in a quantity (either a scalar or a vector). We use the subscript *i* to denote an *initial* value of a quantity, and the subscript *f* to denote the *final* value of a quantity.

Δ (DELTA) IS THE SYMBOL FOR A CHANGE

The symbol Δ (delta) means "final minus initial." If a vector $\vec{r}_i$ denotes the initial position of an object relative to the origin (its position at the beginning of a time interval), and $\vec{r}_f$ denotes the final position of the object, then

$$\Delta\vec{r}=\vec{r}_f-\vec{r}_i$$

$\Delta\vec{r}$ means "change of $\vec{r}$" or $\vec{r}_f-\vec{r}_i$ (displacement).
Δt means "change of t" or t_f-t_i (time interval).

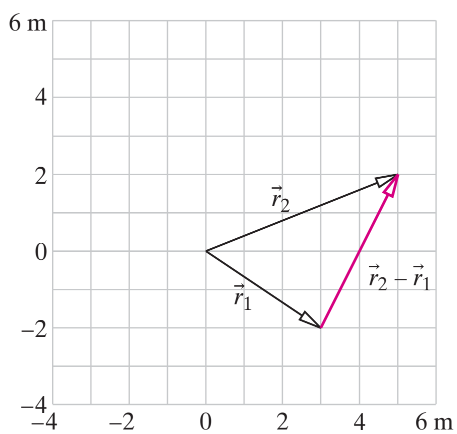

Figure 1.25 Relative position vector.

Since subtraction is not commutative, the order of the quantities matters: the symbol Δ (delta) always means "final minus initial," not "initial minus final." For example, when a child's height changes from 1.1 m to 1.2 m, the change is $\Delta y=+0.1$ m, a positive number. If your bank account dropped from \$150 to \$130, what was the change in your balance? Δ (bank account) = −20 dollars.

Another important application of vector subtraction is the calculation of relative position vectors, vectors that represent the position of one object relative to another object.

RELATIVE POSITION VECTOR

If object 1 is at location $\vec{r}_1$ and object 2 is at location $\vec{r}_2$ (Figure 1.25), the position of 2 relative to 1 is:

$$\vec{r}_{2\text{ relative to }1}=\vec{r}_2-\vec{r}_1$$

Checkpoint 8 At 10:00 AM you are at location $\langle -3,2,5\rangle$ m. By 10:02 AM you have walked to location $\langle 6,4,25\rangle$ m. **(a)** What is $\Delta\vec{r}$, the change in your position? **(b)** What is Δt, the time interval during which your position changed?