Unit Vectors

One way to describe the direction of a vector is by specifying a *unit vector*. A unit vector is a vector of magnitude 1, pointing in some direction. A unit vector is written with a "hat" (caret) over it instead of an arrow. The unit vector $\hat{a}$ is called "a-hat."

QUESTION Is the vector $\langle 1,1,1 \rangle$ a unit vector?

The magnitude of $\langle 1,1,1 \rangle$ is $\sqrt{1^2+1^2+1^2} = 1.73$, so this is not a unit vector.

The vector $\langle 1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3} \rangle$ is a unit vector, since its magnitude is 1:

$$\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2} = 1$$

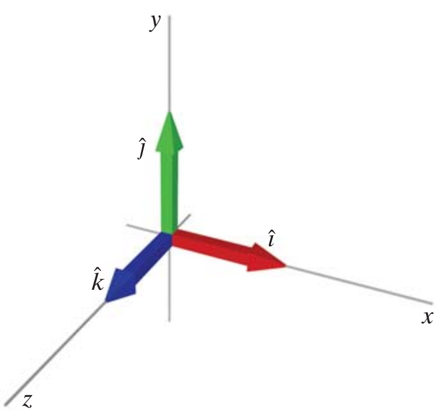

Figure 1.18 The unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

Note that every component of a unit vector must be less than or equal to 1.

In our 3D Cartesian coordinate system, there are three special unit vectors, oriented along the three axes. They are called i-hat, j-hat, and k-hat, and they point along the *x*, *y*, and *z* axes, respectively (Figure 1.18):

$$\hat{\imath} = \langle 1,0,0 \rangle$$
$$\hat{\jmath} = \langle 0,1,0 \rangle$$
$$\hat{k} = \langle 0,0,1 \rangle$$

One way to express a vector is in terms of these special unit vectors:

$$\langle 0.02,-1.7,30.0 \rangle = 0.02\hat{\imath} + (-1.7)\hat{\jmath} + 30.0\hat{k}$$

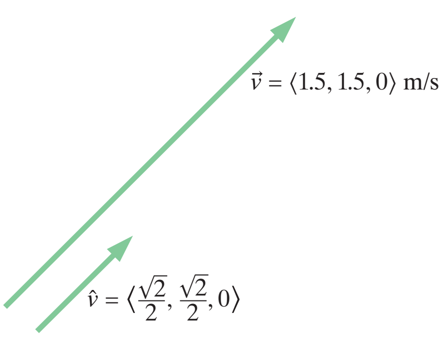

Figure 1.19 The unit vector $\hat{v}$ has the same direction as the vector $\vec{v}$, but its magnitude is 1, and it has no physical units.

Not all unit vectors point along an axis, as shown in Figure 1.19. For example, the vectors

$$\hat{g} = \langle 0.5774, 0.5774, 0.5774 \rangle \quad \text{and} \quad \hat{r} = \langle 0.424, 0.566, 0.707 \rangle$$

are both approximately unit vectors, since the magnitude of each is approximately equal to 1. Again, note that every component of a unit vector is less than or equal to 1.

Any vector may be factored into the product of a unit vector in the direction of the vector, multiplied by a scalar equal to the magnitude of the vector.

$$\vec{w} = |\vec{w}| \cdot \hat{w}$$

For example, a vector of magnitude 5, aligned with the *y* axis, could be written as:

$$\langle 0,5,0 \rangle = 5\langle 0,1,0 \rangle$$

Therefore, to find a unit vector in the direction of a particular vector, we just divide the vector by its magnitude:

FINDING A UNIT VECTOR

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{(x^2 + y^2 + z^2)}}$$

$$\hat{r} = \left\langle \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{y}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \right\rangle$$

EXAMPLE **Magnitude and Direction**

Factor the vector $\vec{v} = \langle -22.3, 0.4, -19.5 \rangle$ m/s into a magnitude times a unit vector.

Solution

$$|\vec{v}| = \sqrt{(-22.3)^2 + (0.4)^2 + (-19.5)^2}\ \text{m/s} = 29.6\ \text{m/s}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -22.3, 0.4, -19.5 \rangle\ \text{m/s}}{29.6\ \text{m/s}} = \langle -0.753, 0.0135, -0.658 \rangle$$

$$\vec{v} = (29.6\ \text{m/s})\langle -0.753, 0.0135, -0.658 \rangle$$

We can now explain algebraically why multiplying a vector by a scalar changes the magnitude but not the direction of a vector. If we write the original vector as the product of a magnitude and a unit vector, after multiplying by a scalar the unit vector is unchanged, but the magnitude is increased or decreased:

$$\begin{aligned}
\vec{a} &= \langle 3, -2, 4 \rangle = (5.385)\langle 0.577, -0.371, 0.743 \rangle \\
2 \cdot \vec{a} &= (2)(5.385)\langle 0.577, -0.371, 0.743 \rangle \\
&= 10.770\langle 0.577, -0.371, 0.743 \rangle \\
&= \langle 6, -4, 8 \rangle
\end{aligned}$$